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PRIME IDEAL, SEMIPRIME IDEAL AND PRIMARY IDEAL OF L-SUBRING

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ABSTRACT

In this paper we introduce the concept of a prime radical of an ideal of an L-ring $L(\mu, R)$ *. Among various results pertaining to this concept, we prove here that prime radicals of an ideal* η*, its radical* $\sqrt{\eta}$ *, its semiprime radical* $S(\eta)$ *and its prime radical* $P(\eta)$ *, all coincide. Also we prove that for a primary ideal, its prime radical coincide with its radical. Moreover, we introduce the concept of primary decomposition and reduced primary decomposition of an ideal in an L-ring. We obtain a necessary and sufficient conditions for an ideal of an L-ring to have a primary decomposition. Some more results pertaining to the decomposition of an ideal are established.*

INTRODUCTION

With this machinery at our disposal, in this paper, we have further introduced the concept of a prime radical of an ideal of an L -ring $L(\mu, R)$. It is proved that the prime radicals of an ideal η, its radical $\sqrt{\eta}$, its semiprime radical $S(\eta)$ and its prime radical $P(\eta)$, are identical. It is also proved that the prime radical of an ideal of an L -ring is always a semiprime ideal. We have also proved that for a primary ideal of an L -ring, its radical, semiprime radical and prime radical coincide. We have established that semiprime radical of the prime radical of an ideal of L -ring is the prime radical of the ideal.

PRIME IDEAL

Definition 2.1 Let $L(\mu, R)$ be any L-ring. An ideal $\eta \neq \mu$ of μ is said to be a *prime* ideal of μ if for all $x, y \in R$, either

$$\eta(xy) \wedge \mu(x) \wedge \mu(y) = \eta(x) \wedge \mu(y) \textit{ or } \eta(xy) \wedge \mu(x) \wedge \mu(y) = \eta(y) \wedge \mu(x) .$$

Theorem 2.2 *Let* $L(\mu, R)$ *be an L-ring. An ideal* η *of* μ *is a prime ideal of* μ *if and only if for each non-empty level subset* η_t*, either* $\eta_t = \mu_t$*, or* η_t *is a prime ideal of* μ_t*.*

Proof. Let η be a prime ideal of μ. Suppose η_t is a non-empty level subset of η such that $\eta_t \neq \mu_t$. η_t is an ideal of μ_t. Let $x, y \in \mu_t$ such that $xy \in \eta_t$. Then $\eta(xy) \geq t, \mu(x) \geq t \text{ and } \mu(y) \geq t$. Since η is a prime ideal of μ, either

$\eta(x) \wedge \mu(y) = \eta(xy) \wedge \mu(x) \wedge \mu(y) \geq t$ or $\eta(y) \wedge \mu(x) = \eta(xy) \wedge \mu(x) \wedge \mu(y) \geq t$.

Thus either $\eta(x) \geq \eta(x) \wedge \mu(y) \geq t$ or $\eta(y) \geq \eta(y) \wedge \mu(x) \geq t$. That is, either $x \in \eta_t$ or $y \in \eta_t$. Hence η_t is a prime ideal of μ_t. Conversely, suppose for each non-empty level subset η_t, either $\eta_t = \mu_t$ or η_t is a prime ideal of μ_t. Let $x, y \in R$. Write $\eta(xy) \wedge \mu(x) \wedge \mu(y) = t$. Then $xy \in \eta_t$, $x \in \mu_t$, $y \in \mu_t$. If $\eta_t = \mu_t$, then $x, y \in \eta_t$. If η_t is a prime ideal of μ_t, then either $x \in \eta_t$ or $y \in \eta_t$. Suppose that $x \in \eta_t$. Then $\eta(x) \geq t$ implies that

$$\eta(x) \wedge \mu(y) \geq t \wedge t = t = \eta(xy) \wedge \mu(x) \wedge \mu(y).$$

Since η is an ideal of μ, we have

$$\eta(xy) \wedge \mu(x) \wedge \mu(y) \geq \eta(x) \wedge \mu(y).$$

Thus $\eta(xy) \wedge \mu(x) \wedge \mu(y) = \eta(x) \wedge \mu(y)$. Similarly if $y \in \eta_t$, then

$$\eta(xy) \wedge \mu(y) \wedge \mu(y) = \eta(y) \wedge \mu(y).$$

Hence η is a prime ideal of μ.■

Theorem 2.3. *Let L be a chain and* $L(\mu, R)$ *be an L-ring. A subring* $\eta \neq \mu$ *of* μ *is a prime ideal of* μ *if and only if, for all* $x, y \in R$

$$\eta(xy) \wedge \mu(x) \wedge \mu(y) = [\eta(x) \wedge \mu(y)] \vee [\eta(y) \wedge \mu(x)].$$

Definition 2.4. Let $L(\mu, R)$ be an L-ring. An ideal $\eta \neq \mu$ of μ is said to be a *semiprime* ideal of μ if

$$\eta(x^n) \wedge \mu(x) = \eta(x), \ \forall x \in R \ \& \ \forall n \in Z^+ .$$

Theorem 2.5. *Let* $L(\mu, R)$ *be an L-ring. Let* $\eta \neq \mu$ *, be an ideal of* μ*. Then* η *is a semiprime ideal of* μ *if and only if, for each non-empty level subset* η_t *either* $\eta_t = \mu_t$ *or* η_t *is a semiprime ideal of* μ_t *.*

Proof. Suppose η is a semiprime ideal of μ. Let η_t be a non-empty level subset such that $\eta_t \neq \mu_t$. Suppose that $x^2 \in \eta_t$ with $x \in \mu_t$. Since η is a semiprime ideal of μ , we have

$$\eta(x) = \eta(x^2) \wedge \mu(x) \geq t \wedge t = t .$$

Hence $x \in \eta_t$. Thus η_t is a semiprime ideal of μ_t.

Conversely, suppose $\eta \neq \mu$ is an ideal of μ such that for each non-empty level subset η_t, either $\eta_t = \mu_t$ or η_t is a semiprime ideal of μ_t. Let $x \in R$, $n \in Z^+$. Write $\eta(x^n) \wedge \mu(x) = t$. Then $\eta(x^n) \geq t$ and $\mu(x) \geq t$. Thus $x^n \in \eta_t$ and $x \in \mu_t$. If $\eta_t = \mu_t$, then $x \in \eta_t$. If η_t is a semiprime ideal of μ_t, then $x \in \eta_t$. Thus $\eta(x) \geq t = \eta(x^n) \wedge \mu(x)$. Since η is an ideal of μ, $\eta(x^n) \geq \eta(x)$. Hence $\eta(x^n) \wedge \mu(x) \geq \eta(x) \wedge \mu(x) = \eta(x)$. Thus $\eta(x^n) \wedge \mu(x) = \eta(x)$, $\forall x \in R, n \in Z^+$. Hence η is a semiprime ideal of μ.■

Theorem 2.6 *Let* $L(\mu, R)$ *be an L-ring and* η *be a prime ideal of* μ *. Then* η *is a semiprime ideal of* μ *.*

Proof. Let $x \in R$. We show that

$$\eta(x^n) \wedge \mu(x) = \eta(x), \qquad \forall n \in Z^+ .$$

We prove the result by induction on n. For n=1, the result is obviously true. Assume that the result is true for n=k. Then $\eta(x^k) \wedge \mu(x) = \eta(x)$. Since η is prime ideal of μ, we have either

$\eta(x^{k+1}) \wedge \mu(x^k) \wedge \mu(x) = \eta(x^k) \wedge \mu(x)$ or $\eta(x^{k+1}) \wedge \mu(x^k) \wedge \mu(x) = \eta(x) \wedge \mu(x^k)$.

Since $L(\mu, R)$ is an L-ring, we have $\mu(x^k) \geq \mu(x) \geq \eta(x)$. Thus

$$\eta(x) \wedge \mu(x^k) = \eta(x) \quad \text{and} \quad \mu(x^k) \wedge \mu(x) = \mu(x).$$

Hence

$$\eta(x^{k+1}) \wedge \mu(x) = \eta(x).$$

Thus η is a semiprime ideal of μ .■

Definition 2.7. Let L be a complete lattice and $L(\mu, R)$ be an L-ring. Let η be an ideal of μ. The *Radical* of η, denoted by $\sqrt{\eta}$, is defined by

$$\sqrt{\eta}(x) = \bigvee_{n \in Z^+} \left[\eta(x^n) \wedge \mu(x)\right] \quad , \forall\, x \in R .$$

Clearly $\eta \subseteq \sqrt{\eta} \subseteq \mu$.

Theorem 2.8. *Let L be a complete lattice and* $L(\mu, R)$ *be an L-ring. An ideal* η *of* μ *is a semiprime ideal of* μ *if and only if* $\sqrt{\eta} = \eta$*.*

Proof. Suppose η is a semiprime ideal of μ. Then

$$\eta(x^n) \wedge \mu(x) = \eta(x) , \quad \forall\, x \in R, \forall n \in Z^+ .$$

Thus

$$\sqrt{\eta}(x) = \bigvee_{n \in Z^+} \left[\eta(x^n) \wedge \mu(x)\right] = \eta(x), \quad \forall x \in R .$$

Hence $\sqrt{\eta} = \eta$.

Conversely, suppose that $\sqrt{\eta} = \eta$. Then $\sqrt{\eta}(x) = \eta(x), \; \forall x \in R$. Hence,

$$\vee_{n\in Z^+}\left[\eta(x^n)\wedge\mu(x)\right]=\eta(x),\quad \forall\, x\in R.$$

Let $m\in Z^+$ and $x\in R$. Then

$$\eta(x)=\vee_{n\in Z^+}\left[\eta(x^n)\wedge\mu(x)\right]\geq\eta(x^m)\wedge\mu(x).$$

Since η is an ideal of μ, $\eta(x^m)\geq\eta(x)$. Thus

$$\eta(x^m)\wedge\mu(x)\geq\eta(x)\wedge\mu(x)=\eta(x).$$

Hence $\eta(x^m)\wedge\mu(x)=\eta(x)$. Therefore η is a semiprime ideal of μ.■

Lemma 2.9. *Let L be a complete lattice and* $L(\mu,R)$ *be an L-ring. Let* η *be an ideal of* μ *and* η *has sup property. Then* $\left(\sqrt{\eta}\right)_t=\sqrt{\eta_t}\cap\mu_t$, $\forall\, t\in L$.

Proof. Let $x\in R$. Since η has sup property, we have $\vee_{n\in Z^+}\eta(x^n)=\eta(x^m)$ for some $m\in Z^+$. Thus

$$\eta(x^m)\wedge\mu(x)=\left[\vee_{n\in Z^+}\eta(x^n)\wedge\right]\mu(x)\geq\eta(x^k)\wedge\mu(x),\qquad \forall k\in Z^+.$$

Hence

$$\eta(x^m)\wedge\mu(x)\geq\vee_{n\in Z^+}\left[\eta(x^n)\wedge\mu(x)\right]\geq\eta(x^m)\wedge\mu(x).$$

Consequently

$$\vee_{n\in Z^+}\left[\eta(x^n)\wedge\mu(x)\right]=\eta(x^m)\wedge\mu(x).$$

Let $x\in\sqrt{\eta_t}\cap\mu_t$. Then $x^k\in\eta_t$ and $x\in\mu_t$ for some $k\in Z^+$. Thus

$$\sqrt{\eta}(x)=\vee_{n\in Z^+}\left[\eta(x^n)\wedge\mu(x)\right]\geq\eta(x^k)\wedge\mu(x)\geq t\wedge t=t.$$

Hence $x\in\left(\sqrt{\eta}\right)_t$ and therefore $\sqrt{\eta_t}\cap\mu_t\subseteq\left(\sqrt{\eta}\right)_t$. To prove the reverse inclusion, let $x\in\left(\sqrt{\eta}\right)_t$. Then $\sqrt{\eta}(x)\geq t$. Hence

$$\eta(x^m) \wedge \mu(x) = \vee_{n \in Z^+} \left[\eta(x^n) \wedge \mu(x)\right] = \sqrt{\eta}(x) \geq t \ .$$

Thus $x^m \in \eta_t$ and $x \in \mu_t$. Consequently $x \in \sqrt{\eta_t} \cap \mu_t$. Therefore $\left(\sqrt{\eta}\right)_t \subseteq \sqrt{n_t} \cap \mu_t$.■

Theorem 2.10. *Let R be a commutative ring and L be a complete lattice. Let* $L(\mu, R)$ *be an L-ring and* η *be an ideal of* μ *and has sup property. Then* $\sqrt{\eta}$ *is an ideal of* μ.

Proof. Let $\left(\sqrt{\eta}\right)_t$ be a non-empty level subset. Let $x, y \in \left(\sqrt{\eta}\right)_t$ and $a \in \mu_t$. Then by Lemma 2.9 $x, y \in \sqrt{\eta_t} \cap \mu_t$. Hence there exist positive integers m and n such that $x^n \in \eta_t$, $y^m \in \eta_t$ and $x, y \in \mu_t$. Now $(-x)^n = x^n$ or $-x^n$. Since η is an ideal of μ, by Theorem 1.6 the non-empty level subset η_t is an ideal of level subring μ_t. Thus $(-x)^n \in \eta_t$. Consequently $-x \in \sqrt{\eta_t}$ and hence $-x \in \left(\sqrt{\eta}\right)_t$. Now

$$(x+y)^{n+m} = x^{n+m} + nx^{m+n-1}y + \dots {}^nC_r x^{n+m-r}y^r + \square + y^{n+m} .$$

Since η_t is an ideal of μ_t, we have ${}^nC_r x^{n+m-r}y^r \in \eta_t$. Hence $(x+y)^{n+m} \in \eta_t$. Consequently $(x+y) \in \sqrt{\eta_t} \cap \mu_t = \left(\sqrt{\eta}\right)_t$. Now $(xa)^n = x^n a^n \in \eta_t$ and hence $xa \in \sqrt{\eta_t} \cap \mu_t = \left(\sqrt{\eta}\right)_t$. Thus $\left(\sqrt{\eta}\right)_t$ is an ideal of μ_t. By Theorem 1.6, $\sqrt{\eta}$ is an ideal of μ.■

Theorem 2.11. *Let R be a commutative ring and L be a complete Heyting algebra. Let* $L(\mu, R)$ *be an L-ring and* η *be an ideal of* μ. *Then* $\sqrt{\eta}$ *is an ideal of* μ.

Proof. Let $x, y \in R$. Clearly $\sqrt{\eta}(-x) = \sqrt{\eta}(x)$. Let $m, n \in Z^+$. Now

$$(x+y)^{n+m} = x^{n+m} + \sum_{i=1}^{n} {}^{n+m}C_i\, x^{n+m-i} y^i + \sum_{i=n+1}^{n+m-1} {}^{n+m}C_i\, x^{n+m-i} y^i + y^{n+m} .$$

Since η is an ideal of μ we have,

$$\eta(x^{n+m-i}y^{i}) \geq \eta(x^{n+m-i}) \wedge \mu(y^{i}), \qquad \forall i = 1,2,\dots, n .$$

$$\eta(x^{n+m-i}y^{i}) \geq \mu(x^{n+m-i}) \wedge \eta(y^{i}), \qquad \forall i = n+1,\dots, n+m-i .$$

Also, $\mu(x+y) \geq \mu(x) \wedge \mu(y)$ as μ is L-ring. Now

$$\eta((x+y)^{n+m}) \geq \eta(x^{n+m}) \wedge \left\{ \bigwedge_{i=1}^{n} \eta(x^{n+m-i}y^{i}) \right\} \wedge \left\{ \bigwedge_{i=n+1}^{n+m-1} \eta(x^{n+m-i}y^{i}) \right\} \wedge \eta(y^{n+m}) .$$

Therefore

$$\eta((x+y)^{n+m}) \wedge \mu(x+y) \geq \eta(x^{n+m}) \wedge \left\{ \bigwedge_{i=1}^{n} \eta\left((x^{n+m-i}) \wedge \mu(y^{i}) \right) \right\} \wedge$$
$$\left\{ \bigwedge_{i=n+1}^{n+m-1} \left(\mu(x^{n+m-i}) \wedge \eta(y^{i}) \right) \right\} \wedge \eta(y^{m+n}) \wedge \left(\mu(x) \wedge \mu(y) \right)$$
$$= \left\{ \bigwedge_{i=m}^{m+n} \left(\eta(x^{i}) \wedge \mu(x) \right) \right\} \wedge \left\{ \bigwedge_{i=n+1}^{m+n} (\eta(y^{i}) \wedge \mu(y)) \right\}$$

(Since $\mu(x^{i}) \geq \mu(x), \forall i = 1,2,\dots$).

Again, since η is an ideal of μ, we have

$$\eta(x^{m+1}) \wedge \mu(x) \geq \eta(x^{m}) \wedge \mu(x) \wedge \mu(x) = \eta(x^{m}) \wedge \mu(x) .$$

From this it follows that $\eta(x^{m+k}) \wedge \mu(x) \geq \eta(x^{m}) \wedge \mu(x)$, $\forall k \in Z^{+}$. Thus

$$\bigwedge_{i=m}^{m+n} \left(\eta(x^{i}) \wedge \mu(x) \right) = \eta(x^{m}) \wedge \mu(x).$$

Similarly

$$\bigwedge_{i=n+1}^{m+n} \left(\eta(y^{i}) \wedge \mu(y) \right) = \eta(y^{n+1}) \wedge \mu(y) \geq \eta(y^{n}) \wedge \mu(y) .$$

Therefore

$$\eta((x+y)^{n+m}) \wedge \mu(x+y) \geq \left[\eta(x^{m}) \wedge \mu(x) \right] \wedge \left[\eta(y^{n}) \wedge \mu(y) \right] .$$

Now

$$\sqrt{\eta}(x+y) = \underset{k\in Z^+}{\vee}\left[\eta(x+y)^k \wedge \mu(x+y)\right]$$

$$\geq \Big((x+y)^{n+m}\Big) \wedge \mu(x+y)$$

$$\geq \left\{\eta(x^m) \wedge \mu(x)\right\} \wedge \left\{\eta(y^n) \wedge \mu(x)\right\},\ \forall m, n \in Z^+ .$$

Thus for an arbitrary but fixed $n \in Z^+$, we have

$$\sqrt{\eta}(x+y) \geq \underset{m\in Z^+}{\vee}\left\{\Big(\eta(x^m) \wedge \mu(x)\Big) \wedge \Big(\eta(y^n) \wedge \mu(y)\Big)\right\}$$

$$= \left\{\underset{m\in Z^+}{\vee}\Big(\eta(x^m) \wedge \mu(x)\Big)\right\} \wedge \Big(\eta(y^n) \wedge \mu(y)\Big)$$

(Since L is complete Heyting algebra)

$$= \sqrt{\eta}(x) \wedge \Big(\eta(y^n) \wedge \mu(y)\Big).$$

Again since L is complete Heyting algebra and n is arbitrary, we have

$$\sqrt{\eta}(x+y) \geq \underset{n\in Z^+}{\vee}\left\{\sqrt{\eta}(x) \wedge \Big(\eta(y^n) \wedge \mu(y)\Big)\right\}$$

$$= \sqrt{\eta}(x) \wedge \left\{\underset{n\in Z^+}{\vee}\Big(\eta(y^n) \wedge \mu(y)\Big)\right\}$$

$$= \sqrt{\eta}(x) \wedge \sqrt{\eta}(y).$$

Now

$$\eta((xy)^n) \wedge \mu(xy) \geq \eta(x^n y^n) \wedge \mu(x) \wedge \mu(y)$$

$$\geq \eta(x^n) \wedge \mu(y^n) \wedge \mu(x) \wedge \mu(y) \quad \text{(Since } \eta \text{ is an ideal of } \mu\text{)}$$

$$= \Big(\eta(x^n) \wedge \mu(x)\Big) \wedge \mu(y) \qquad \text{(Since } \mu(y^n) \geq \mu(y)\text{)}$$

Therefore

$$\sqrt{\eta}(xy) = \underset{n\in Z^+}{\vee}\left\{\eta((xy)^n) \wedge \mu(xy)\right\}$$

$$\geq \bigvee_{n\in Z^+} \left\{ \left(\eta(x^n) \wedge \mu(x)\right) \wedge \mu(y) \right\}$$

$$= \left\{ \bigvee_{n\in Z^+} \left(\eta(x^n) \wedge \mu(x)\right) \right\} \wedge \mu(y)$$

(Since L is complete Heyting algebra)

$$= \sqrt{\eta}(x) \wedge \mu(y)$$

Similarly $\sqrt{\eta}(xy) \geq \sqrt{\eta}(y) \wedge \mu(x)$. Hence $\sqrt{\eta}$ is an ideal of μ .■

Theorem 2.12. *Let L be a complete lattice and* $\mathrm{L}(\mu, \mathrm{R})$ *be an L-ring. Let* η *and* θ *be ideals of* μ*. Then*

$$\eta \subseteq \theta \Rightarrow \sqrt{\eta} \subseteq \sqrt{\theta}\,.$$

Proof. Obvious.

Theorem 2.13. *Let R be a commutative ring and L be a complete Heyting algebra. Let* $L(\mu, R)$ *be an L-ring and* η *be an ideal of* μ*. Then* $\sqrt{\sqrt{\eta}} = \sqrt{\eta}$ *.*

Proof. By Theorem 2.11, $\sqrt{\eta}$ is an ideal of μ . Since $\eta \subseteq \sqrt{\eta}$, by the above theorem, we have $\sqrt{\eta} \subseteq \sqrt{\sqrt{\eta}}$. To prove the reverse inclusion, let $x \in R$. Now

$$\sqrt{\sqrt{\eta}}(x) = \bigvee_{n\in Z^+} \left\{ \sqrt{\eta}(x^n) \wedge \mu(x) \right\}$$

$$= \bigvee_{n\in Z^+} \left\{ \bigvee_{m\in Z^+} \left[\eta(x_{nm}) \wedge \mu(x_n)\right] \wedge \mu(x) \right\}$$

$$= \bigvee_{n\in Z^+} \left\{ \bigvee_{m\in Z^+} \left[\eta(x_{nm}) \wedge \mu(x_n) \wedge \mu(x)\right] \right\} .$$

(Since L is complete Heyting algebra)

$$= \bigvee_{n\in Z^+} \left\{ \bigvee_{m\in Z^+} \left[\eta(x^{nm}) \wedge \mu(x)\right] \right\}. \qquad \text{(Since } \mu(x^n) \geq \mu(x)\text{)}$$

Since for each $n \in Z^+$, $\underset{m \in Z^+}{\vee}\left[\eta(x^{nm}) \wedge \mu(x)\right] \leq \sqrt{\eta}(x)$, we have

$$\sqrt{\sqrt{\eta}}(x) = \underset{n \in Z^+}{\vee}\left\{\underset{m \in Z^+}{\vee}\left[\eta(x^{nm}) \wedge \mu(x)\right]\right\} \leq \sqrt{\eta}(x).$$

Thus $\sqrt{\sqrt{\eta}} \subseteq \sqrt{\eta}$. Consequently $\sqrt{\sqrt{\eta}} = \sqrt{\eta}$.■

Theorem 2.18. *Let R be a commutative ring and L be a completely distributive lattice. Let* $L(\mu, R)$ *be an L-ring. Let* η *and* θ *be ideals of* μ*. Then* $\sqrt{\eta \cap \theta} = \sqrt{\eta} \cap \sqrt{\theta} = \sqrt{\eta\theta}$ *.*

Proof. Let $x \in R$. Now

$$\sqrt{\eta \cap \theta}(x) = \underset{n \in Z^+}{\vee}\left[(\eta \cap \theta)(x^n) \wedge \mu(x)\right] = \underset{n \in Z^+}{\vee}\left[\eta(x^n) \wedge \theta(x^n) \wedge \mu(x)\right]$$

$$= \underset{n \in Z^+}{\vee}\left\{\left[\eta(x^n) \wedge \mu(x)\right] \wedge \left[\theta(x^n) \wedge \mu(x)\right]\right\}$$

$$= \left\{\underset{n \in Z^+}{\vee}\left[\eta(x^n) \wedge \mu(x)\right]\right\} \wedge \left\{\underset{n \in Z^+}{\vee}\left[\theta(x^n) \wedge \mu(x)\right]\right\}$$

(Since L is completely distributive)

$$= \sqrt{\eta}(x) \wedge \sqrt{\theta}(x) = \left(\sqrt{\eta} \cap \sqrt{\theta}\right)(x) .$$

Thus $\sqrt{\eta \cap \theta} = \sqrt{\eta} \cap \sqrt{\theta}$.

Now, since η and θ are ideals of μ , By Theorem 2.17, $\eta\theta$ is an ideal of μ . Also by Theorem 2.15, we have $\eta\theta \subseteq \eta\mu \subseteq \eta$. Therefore by Theorem 2.12, $\sqrt{\eta\theta} \subseteq \sqrt{\eta}$. Similarly $\sqrt{\eta\theta} \subseteq \sqrt{\theta}$. Thus $\sqrt{\eta\theta} \subseteq \sqrt{\eta} \cap \sqrt{\theta} = \sqrt{\eta \cap \theta}$. Next, let $x \in R$. Then

$$\sqrt{\eta\theta}(x) = \underset{n \in Z^+}{\vee}\left[(\eta\theta)(x^n) \wedge \mu(x)\right] \geq \underset{n \geq 2}{\vee}\left\{\left[\underset{r=1}{\overset{n-1}{\vee}}\left(\eta(x^r) \wedge \theta(x^{n-r})\right)\right] \wedge \mu(x)\right\} .$$

Now

$$\bigvee_{r=1}^{n-1}\left[\eta(x^{r})\wedge\theta(x^{n-r})\right]\geq\left[\eta(x^{n-1})\wedge\theta(x)\right]\vee\left[\eta(x)\wedge\theta(x^{n-1})\right]$$

$$=\left[\eta(x^{n-1})\vee\eta(x)\right]\wedge\left[\theta(x^{n-1})\vee\theta(x)\right]$$

(Since L is completely distributive)

$= \eta(x^{n-1})\wedge\theta(x^{n-1})$ (Since $\eta(x^{n-1})\geq\eta(x)$)

$= (\eta\cap\theta)(x^{n-1})$.

Thus $\sqrt{\eta\theta}(x)\geq\bigvee_{n\geq 2}\left\{(\eta\cap\theta)(x^{n-1})\wedge\mu(x)\right\}=\sqrt{(\eta\cap\theta)}(x)$. Hence $\sqrt{\eta\cap\theta}\subseteq\sqrt{\eta\theta}$.

Consequently $\sqrt{\eta\theta}=\sqrt{\eta\cap\theta}$.■

Theorem 2.19. *Let R be a commutative ring and L be a complete Heyting algebra. Let* $L(\mu, R)$ *be an L-ring. Let* η *and* θ *be ideals of* μ *with* $\eta(0)=\theta(0)$ *. Then*

$$\sqrt{\eta}+\sqrt{\theta}\subseteq\sqrt{\sqrt{\eta}+\sqrt{\theta}}=\sqrt{\eta+\theta}\ .$$

Proof. By Theorem 2.11, $\sqrt{\eta}$ and $\sqrt{\theta}$ are ideals of μ . By Theorem 2.16, $\eta+\theta$ and $\sqrt{\eta}+\sqrt{\theta}$ are an ideals of μ . Clearly $\sqrt{\eta}+\sqrt{\theta}\subseteq\sqrt{\sqrt{\eta}+\sqrt{\theta}}$. Since $\eta\subseteq\sqrt{\eta}$ and $\theta\subseteq\sqrt{\theta}$, we have $\eta+\theta\subseteq\sqrt{\eta}+\sqrt{\theta}$. Thus by Theorem 2.12 $\sqrt{\eta+\theta}\subseteq\sqrt{\sqrt{\eta}+\sqrt{\theta}}$. By Theorem 2.11, $\sqrt{\eta+\theta}$ is an ideal of μ . Thus, $\sqrt{\eta+\theta}+\sqrt{\eta+\theta}=\sqrt{\eta+\theta}$. Since $\eta\subseteq\eta+\theta$, by Theorem 2.12 we have $\sqrt{\eta}\subseteq\sqrt{\eta+\theta}$. Similarly $\sqrt{\theta}\subseteq\sqrt{\eta+\theta}$. Therefore

$$\sqrt{\eta}+\sqrt{\theta}\subseteq\sqrt{\eta+\theta}+\sqrt{\eta+\theta}=\sqrt{\eta+\theta}\ .$$

Thus by Theorem 2.12 and Theorem 2.13, $\sqrt{\sqrt{\eta}+\sqrt{\theta}}\subseteq\sqrt{\sqrt{\eta+\theta}}=\sqrt{\eta+\theta}$. Hence $\sqrt{\sqrt{\eta}+\sqrt{\theta}}=\sqrt{\eta+\theta}$.■

Definition 2.20. *Let* $L(\mu, R)$ *be an L-ring. An ideal* $\eta \neq \mu$ *of* μ *is said to be primary ideal of* μ *if for all* $x, y \in R$*, we have either*

$$\eta(x) \wedge \mu(y) \geq \eta(xy) \wedge \mu(x) \wedge \mu(y) \qquad (1.1)$$

or

$$\eta(y) \wedge \mu(x) \geq \eta(xy) \wedge \mu(x) \wedge \mu(y) \qquad (1.2)$$

or

$$\eta(x^n) \wedge \mu(x) \wedge \eta(y^m) \wedge \mu(y) \geq \eta(xy) \wedge \mu(x) \wedge \mu(y), \qquad (1.3)$$

for some integers m,n > 1.

Obviously, every prime ideal of an L-ring $L(\mu, R)$ is a primary ideal of μ.

Lemma 2.21. *Let R be a ring. An ideal I of R is primary if and only if, whenever* $xy \in I$ *we have either*

$x \in I$ *or* $y \in I$ *or (*x^n *&* $y^m \in I$ *), for some integers m,n>1.*

Proof. Suppose that the ideal I is primary. Let $xy \in I$. Then we consider the following three cases.

case (i) $x \notin I, y \notin I$.

Since I is a primary ideal and $x \notin I$, we have $y^m \in I$ for some positive integer m. Also $m > 1$, since $y \notin I$. Similarly, we have $x^n \in I$ for some integer n > 1.

Case (ii) $x \notin I$ and either $x^n \notin I$ or $y^n \notin I$ for any integer $n > 1$.

Again, since I is a primary ideal and $x \notin I$, we have $y^m \in I$ for some integer $m \geq 1$. We show that $y \in I$. Suppose $y \notin I$. Then m > 1. Therefore by the hypothesis $x^n \notin I$ for any integer n > 1. Since I is primary and $y \notin I$, $x^m \in I$ for some integer $m \geq 1$. As $x \notin I$, therefore m > 1. Hence $x^m \in I$ for some integer m > 1, which is a contradiction. Thus $y \in I$.

Case(iii) $y \notin I$ and either $x^n \notin I$ or $y^n \notin I$ for any integer n > 1. The proof of this part is similar to that of case (ii).

To prove the converse part, suppose $xy \in I$ and $x \notin I$. Then either $y \in I$ or there exists integers $m, n > 1$ such that $x^n \in I$ and $y^m \in I$. Thus, in either case $y^m \in I$ for some positive integer m. Similarly if $y \notin I$, then $x^n \in I$ for some positive integer n. Thus I is a primary ideal of R. ■

Theorem 2.22. *Let* $L(\mu, R)$ *be an L-ring and* η *be an ideal of* μ *with* $\eta \neq \mu$ *. Then* η *is a primary ideal of* μ *if and only if for each non-empty level subset* η_t*, either* $\eta_t = \mu_t$ *or* η_t *is a primary ideal of* μ_t*.*

Proof. Suppose η is a primary ideal of μ and η_t is a non-empty level subset such that $\eta_t \neq \mu_t$. Let $xy \in \eta_t$, $x, y \in \mu_t$. Then it follows that $\eta(xy) \wedge \mu(x) \wedge \mu(y) \geq t$. Since η is primary ideal of μ, one of the conditions (1.1), (1.2) and (1.3) hold. Now, if condition (1.1) holds then

$$\eta(x) \geq \eta(x) \wedge \mu(y) \geq \eta(xy) \wedge \mu(x) \wedge \mu(y) \geq t .$$

Thus $x \in \eta_t$. If (1.2) holds, then

$$\eta(y) \geq \eta(y) \wedge \mu(x) \geq \eta(xy) \wedge \mu(x) \wedge \mu(y) \geq t .$$

Thus $y \in \eta_t$. In case condition (1.3) is valid, we have

$$\eta(x^n) \wedge \mu(x) \wedge \eta(y^m) \wedge \mu(y) \geq \eta(xy) \wedge \mu(x) \wedge \mu(y) \geq t$$

for some integer m,n>1.

Thus $x^n, y^m \in \eta_t$, Therefore, by Lemma 2.21, η_t is a primary ideal of μ_t.

Our next result shows that every semiprime ideal of an L-ring which is also primary is a prime ideal. ■

Theorem 2.23. *Let* $L(\mu, R)$ *be an L-ring and* η *be a semiprime ideal of* μ. *If* η *is a primary ideal of* μ, *then* η *is a prime ideal of* μ.

Proof. Let $x, y \in R$. Since η is semiprime ideal of μ, we have

$$\eta(x^n) \wedge \mu(x) = \eta(x) \text{ and } \eta(y^m) \wedge \mu(y) = \eta(y), \ \forall n, m \in Z^+.$$

Thus

$$\eta(x^n) \wedge \mu(x) \wedge \eta(y^m) \wedge \mu(y) = \eta(x) \wedge \eta(y), \ \forall n, m \in Z^+ \quad (1.4)$$

Since η is a primary ideal of μ, one of the conditions (1.1), (1.2) and (1.3) holds. If condition (1.3) holds then for some integers r,s > 1, we have

$$\eta(x^r) \wedge \mu(x) \wedge \eta(y^s) \wedge \mu(y) \geq \eta(xy) \wedge \mu(x) \wedge \mu(y).$$

From this alongwith (1.4), we have

$$\eta(x) \wedge \mu(y) \geq \eta(x) \wedge \eta(y) = \eta(x^r) \wedge \mu(x) \wedge \eta(y^s) \wedge \mu(y)$$

$$\geq \eta(xy) \wedge \mu(x) \wedge \mu(y).$$

This again gives us condition (1.1). Therefore, either condition (1.1) or (1.2) holds. Since η is an ideal of μ, by Lemma 1.17, we have

$$\eta(xy) \wedge \mu(x) \wedge \mu(y) \geq \eta(x) \wedge \mu(y) \text{ and } \eta(xy) \wedge \mu(x) \wedge \mu(y) \geq \eta(y) \wedge \mu(x).$$

Thus either,

$$\eta(xy) \wedge \mu(x) \wedge \mu(y) = \eta(x) \wedge \mu(y) \text{ or } \eta(xy) \wedge \mu(x) \wedge \mu(y) = \eta(y) \wedge \mu(x).$$

Therefore η is a prime ideal of μ.■

Theorem 2.24. *Let R be a commutative ring and L be a complete lattice. Let* $L(\mu, R)$ *be an L-ring and* η *be a primary ideal of* μ *and has sup property. Then* $\sqrt{\eta}$ *is a prime ideal of* μ. *Also* $\sqrt{\sqrt{\eta}} = \sqrt{\eta}$.

Proof. By Theorem 2.10, $\sqrt{\eta}$ is an ideal of μ .Let $x, y \in R$. Since η has sup property, there exists $m \in Z^+$ such that

$$\sqrt{\eta}(xy) = \underset{n \in Z^+}{\vee} \left[\eta(xy)^n \wedge \mu(xy)\right] = \eta(x^m y^m) \wedge \mu(xy). \qquad (1.5)$$

Now

$$\sqrt{\eta}(x) = \underset{n \in Z^+}{\vee} \left[\eta(x^n) \wedge \mu(x)\right] \geq \eta(x^s) \wedge \mu(x), \quad \forall s \in Z^+.$$

Hence

$$\sqrt{\eta}(x) \wedge \mu(y) \geq \eta(x^s) \wedge \mu(x) \wedge \mu(y), \quad \forall s \in Z^+. \qquad (1.6)$$

Similarly

$$\sqrt{\eta}(y) \wedge \mu(x) \geq \eta(y^s) \wedge \mu(x) \wedge \mu(y), \quad \forall s \in Z^+ \qquad (1.7)$$

Since η is a primary ideal of μ, by Definition 2.20, we have either

$$\eta(x^m y^m) \wedge \mu(x^m) \wedge \mu(y^m) \leq \eta(x^m) \wedge \mu(y^m) \qquad (1.8)$$

or $$\eta(x^m y^m) \wedge \mu(x^m) \wedge \mu(y^m) \leq \eta(y^m) \wedge \mu(x^m) \qquad (1.9)$$

or $$\eta(x^m y^m) \wedge \mu(x^m) \wedge \mu(y^m) \leq \eta(x^{mk}) \wedge \mu(x^m) \wedge \eta(y^{mr}) \wedge \mu(y^m) \qquad (1.10)$$

for some integers k,r > 1.

By (1.5), we have

$$\begin{aligned}\sqrt{\eta}(xy) \wedge \mu(x) \wedge \mu(y) &= \eta(x^m y^m) \wedge \mu(xy) \wedge \mu(x) \wedge \mu(y) \\ &= \eta(x^m y^m) \wedge \mu(x) \wedge \mu(y) \\ &= \eta(x^m y^m) \wedge \mu(x^m) \wedge \mu(y^m) \wedge \mu(x) \wedge \mu(y).\end{aligned}$$

If (1.8) holds, then

$$\begin{aligned}\sqrt{\eta}(xy) \wedge \mu(x) \wedge \mu(y) &\leq \eta(x^m) \wedge \mu(y^m) \wedge \mu(x) \wedge \mu(y) \\ &= \left[\eta(x^m) \wedge \mu(x)\right] \wedge \mu(y)\end{aligned}$$

$$\leq \sqrt{\eta}(x) \wedge \mu(y). \qquad \text{(by (1.6))}$$

If (1.9) holds, then

$$\sqrt{\eta}(xy) \wedge \mu(x) \wedge \mu(y) \leq \eta(y^m) \wedge \mu(x^m) \wedge \mu(x) \wedge \mu(y)$$

$$= [\eta(y^m) \wedge \mu(x)] \wedge \mu(y)$$

$$\leq \sqrt{\eta}(y) \wedge \mu(y). \qquad \text{(by (1.7))}$$

In case, condition (1.10) is valid, then

$$\sqrt{\eta}(xy) \wedge \mu(x) \wedge \mu(y) \leq \eta(x^{mk}) \wedge \mu(x^m) \wedge \eta(y^{mr}) \wedge \mu(y^m) \wedge \mu(x) \wedge \mu(y)$$

$$= \eta(x^{mk}) \wedge \eta(y^{mr}) \wedge \mu(x) \wedge \mu(y)$$

$$= \left[\eta(x^{mk}) \wedge \mu(x) \wedge \mu(y)\right] \wedge \left[\eta(y^{mr}) \wedge \mu(y) \wedge \mu(x)\right]$$

$$\leq \left[\sqrt{\eta}(x) \wedge \mu(y)\right] \wedge \left[\sqrt{\eta}(y) \wedge \mu(x)\right]$$

$$\leq \sqrt{\eta}(x) \wedge \mu(y).$$

Hence $\sqrt{\eta}$ is a prime ideal of μ. By Theorem 2.6, $\sqrt{\eta}$ is a semiprime ideal and hence by Theorem 2.8, $\sqrt{\sqrt{\eta}} = \sqrt{\eta}$.■

Definition 2.25. *Let R be a commutative ring and L be a complete lattice. Let $L(\mu, R)$ be an L-ring. Let η be a primary ideal of μ and η has sup property . Then $\sqrt{\eta}$ is a prime ideal of μ, called the associated prime ideal of η.*

Our next result shows that the associated prime ideal of η is the smallest prime ideal of μ containing η.

Theorem 2.26. *Let R be a commutative ring and L be a complete lattice. Let $L(\mu, R)$ be an L-ring. Let η be a primary ideals of μ and has sup property. Then the associated prime ideal of η is the smallest prime ideal of μ containing η.*

Proof. Suppose θ is a prime ideal of μ such that $\eta \subseteq \theta$. Since θ is a prime ideal of μ, θ is a semiprime ideal of μ. Hence by Theorem 2.8, we have $\sqrt{\theta} = \theta$. Now $\eta \subseteq \theta$ implies that $\sqrt{\eta} \subseteq \sqrt{\theta} = \theta$.■

Theorem 2.27. *Let R be a commutative ring and L be a complete chain. Let* $L(\mu, R)$ *be an L-ring. Let* η *and* θ *be ideals of* μ *such that* $\eta \subseteq \theta \subseteq \sqrt{\eta}$ *. Suppose that* η *has sup property and for* $a, b \in R$ *, we have*

$$\eta(ab) > \theta(a) \Rightarrow \eta(ab) = \eta(b) .$$

Then η *is a primary ideal of* μ *and* $\sqrt{\eta} = \theta$*.*

Proof. By Theorem 2.10, $\sqrt{\eta}$ is an ideal of μ . Let $a, b \in R$. Then the following three cases arise.

Case (i) $\eta(ab) > \theta(a)$.

Then $\eta(ab) = \eta(b)$. Now $\eta(b) \wedge \mu(a) = \eta(b) \wedge \mu(b) \wedge \mu(a) = \eta(ab) \wedge \mu(a) \wedge \mu(b)$.

Case (ii) $\eta(ab) \le \theta(a)$ and $\eta(ab) > \theta(b)$ **.**

Then $\eta(ab) = \eta(a)$. Now, we have

$$\eta(a) \wedge \mu(b) = \eta(a) \wedge \mu(a) \wedge \mu(b) = \eta(ab) \wedge \mu(a) \wedge \mu(b) .$$

Case (iii) $\eta(ab) \le \theta(a)$ and $\eta(ab) \le \theta(b)$**.**

Since $\eta \subseteq \theta \subseteq \sqrt{\eta}$, we have

$$\eta(ab) \le \theta(a) \le \sqrt{\eta}(a) = \bigvee_{n \in Z^+} \left[\eta(a^n) \wedge \mu(a)\right] = \eta(a^k) \wedge \mu(a) \text{, for some } k \in Z^+$$

(Since η has sup property).

Similarly $\eta(ab) \le \eta(b^m) \wedge \mu(b)$, for some $m \in Z^+$. Thus

$$\eta(ab) \wedge \mu(a) \wedge \mu(b) = \left[\eta(ab) \wedge \mu(a) \wedge \mu(b)\right] \wedge \left[\eta(ab) \wedge \mu(a) \wedge \mu(b)\right]$$

$$\leq \left[\eta(a^k) \wedge \mu(a) \wedge \mu(b)\right] \wedge \left[\eta(b^m) \wedge \mu(a) \wedge \mu(b)\right]$$

$$= \eta(a^k) \wedge \mu(a) \wedge \eta(b^m) \wedge \mu(b)$$, for some $k, m \in Z^+$.

Thus η is a primary ideal of μ.

To show that $\sqrt{\eta} = \theta$, it is sufficient to show that $\sqrt{\eta} \subseteq \theta$. Let $a \in R$. Firstly we show that $\eta(a^n) \leq \theta(a), \ \forall, n \in Z^+$. Suppose this is not the case. Then there exists $k \in Z^+$ such that $\eta(a^k) > \theta(a)$. Let m be the smallest positive integer such that $\eta(a^m) > \theta(a)$. Since $\eta \subseteq \theta$, we have $\eta(a) \leq \theta(a)$. Thus $m \geq 2$. Now $\eta(a^{m-1}a) > \theta(a)$. By the given hypothesis, we have $\eta(a^m) = \eta(a^{m-1})$. Thus $\eta(a^{m-1}) = \eta(a^m) > \theta(a)$, which is a contradiction. So that $\eta(a^n) \leq \theta(a)$, $\forall n \in Z^+$. Therefore

$$\sqrt{\eta}(a) = \vee_{n \in Z^+} \left[\eta(a^n) \wedge \mu(a)\right] \leq \vee_{n \in Z^+} \eta(a^n) \leq \theta(a).$$

Hence $\sqrt{\eta} \subseteq \theta$.■

Lemma 2.28. *Let L be a complete lattice and* $L(\mu, R)$ *be an L-ring. Let* $\{\eta_i\}$ *be a chain of prime ideals of* μ*. Then* $\bigcap_i \eta_i$ *is a prime ideal of* μ*.*

Proof. $\bigcap_i \eta_i$ is an ideal of μ. Let $\left(\bigcap_i \eta_i\right)_t$ be a non-empty level subset. Suppose that $\left(\bigcap_i \eta_i\right)_t \neq \mu_t$. By Lemma 1.14, $\left(\bigcap_i \eta_i\right)_t = \bigcap_i (\eta_i)_t$. Since $\left(\bigcap_i \eta_i\right)_t$ is non-empty, $(\eta_i)_t$ is non-empty for each i. Since for each i, η_i is prime ideal of μ, by Theorem 2.2, either $(\eta_i)_t = \mu_t$ or $(\eta_i)_t$ is a prime ideal of μ_t. Let $xy \in \left(\bigcap_i \eta_i\right)_t$, $x, y \in \mu_t$. Then $xy \in (\eta_i)_t$ for each i. If possible, $x \notin \bigcap_i (\eta_i)_t$ and $y \notin \bigcap_i (\eta_i)_t$. Then there exists j,k such that $x \notin (\eta_j)_t$ and $y \notin (\eta_k)_t$. Since $\{\eta_i\}$ is a chain, we assume that $\eta_j \subseteq \eta_k$. Thus $(\eta_j)_t \subseteq (\eta_k)_t$. Hence $x \notin (\eta_j)_t$ and $y \notin (\eta_j)_t$. This contradicts that either $(\eta_j)_t = \mu_t$ or $(\eta_j)_t$ is a prime ideal of μ_t. Thus, either $x \in \bigcap_i (\eta_i)_t = \left(\bigcap_i \eta_i\right)_t$ or $y \in \bigcap_i (\eta_i)_t$. Hence $\left(\bigcap_i \eta_i\right)_t$ is a prime ideal of μ_t. By Theorem 2.2, η is a prime ideal of μ.

Theorem 2.29. *Let L be a complete lattice and $L(\mu, R)$ be an L-ring. Then, the intersection of an arbitrary family of semiprime ideals of μ is a semiprime ideal of μ.*

Proof. Let $\{\eta_i\}_{i\in\lambda}$ be a family of semiprime ideals of μ. Then by Lemma 1.16, $\bigcap_{i\in\lambda}\eta_i$ is an ideal of μ. Let $x \in R, n \in Z^+$. Since for each $i \in \lambda$, η_i is semiprime ideal of μ, we have

$$\eta_i(x^n) \wedge \mu(x) = \eta_i(x), \quad \forall\ i \in \lambda .$$

Now

$$\left(\bigcap_{i\in\lambda}\eta_i\right)(x^n) \wedge \mu(x) = \left(\bigwedge_{i\in\lambda} \eta_i(x^n)\right) \wedge \mu(x) = \bigwedge_{i\in\lambda}\{\eta_i(x^n) \wedge \mu(x)\}$$

$$= \bigwedge_{i\in\lambda} \eta_i(x) = \left(\bigcap_{i\in\lambda}\eta_i\right)(x) .$$

Thus $\bigcap_{i\in\lambda}\eta_i$ is a semiprime ideal of μ. ■

Definition 2.30. *Let $L(\mu, R)$ be an L-ring and η be an ideal of μ. A prime ideal θ of μ is said to be a minimal prime ideal of η (or an isolated prime ideal of η), if $\eta \subseteq \theta$ and there is no prime ideal ν of μ such that $\eta \subseteq \nu \subsetneq \theta$.*

Let $L(\mu, R)$ be an L-ring. Consider the L-subset $\theta_\mu : R \to L$ defined by

$$\theta_\mu(x) = \wedge_{x\in R} \mu(x), \ \forall\, x \in R.$$

Then θ_μ is a prime ideal of μ called the minimal prime ideal of μ.

Theorem 2.31. *Let L be a complete lattice and* $L(\mu, R)$ *be an L-ring. Let* η *be an ideal of* μ *and* θ *be a prime ideal of* μ *such that* $\eta \subseteq \theta$ *. Then there exists a minimal prime ideal* ν^* *of* η *such that* $\eta \subseteq \nu^* \subseteq \theta$ *.*

Proof. Let $\Im = \{\nu \mid \nu \text{ is prime ideal of } \mu \text{ and } \eta \subseteq \nu \subseteq \theta\}$. The family $\Im$ is non-empty, since $\theta \in \Im$. Define a relation of partial ordering $\leq$ on $\Im$, as follows

$$\nu_1 \leq \nu_2 \text{ if } \nu_2 \subseteq \nu_1 .$$

Consider a chain τ in $\Im$. Write $\nu_0 = \bigcap_{\nu_i \in \tau} \nu_i$. By Lemma 2.28, ν_0 is a prime ideal of μ. Since $\eta \subseteq \nu_i \subseteq \theta$ for all $\nu_i \in \tau$, we have $\eta \subseteq \nu_0 \subseteq \theta$.Thus $\nu_0 \in \Im$. Also $\nu_0 \subseteq \nu_i$ for all $\nu_i \in \tau$. Therefore $\nu_i \leq \nu_0$ for all $\nu_i \in \tau$. Hence ν_0 is an upper bound of the chain τ in $\Im$. By Zorn's Lemma, $\Im$ has a maximal element ν^* (say). That is, $\nu^* \in \Im$ and if $\nu' \in \Im$ with $\nu^* \leq \nu'$, then $\nu^* = \nu'$. Since $\nu^* \in \Im$, ν^* is a prime ideal of μ such that $\eta \subseteq \nu^* \subseteq \theta$. To show that ν^* is a minimal prime ideal of η, let ξ be any prime ideal of μ such that $\eta \subseteq \xi \subseteq \nu^*$. Then $\xi \in \Im$ and $\nu^* \leq \xi$. Since ν^* is maximal element of $\Im$, we have $\nu^* = \xi$ Hence ν^* is a minimal prime ideal of η such that $\eta \subseteq \nu^* \subseteq \theta$.■

Theorem 2.32. *Let R be a commutative ring and* $L(\mu, R)$ *be an L-ring. Let* η *be a semiprime ideal of* ν *and* ν *be an ideal of* μ*. Then* η *is an ideal of* μ*.*

Proof. Let $x, y \in R$. Since η is a semiprime ideal of ν, we have

$$\eta(xy) = \eta((xy)\,(xy)) \wedge \nu(xy)$$

$\geq \eta(x) \wedge \nu(y(xy)) \wedge \nu(xy)$ (Since η is an ideal of ν)

$\geq \eta(x) \wedge \nu(xy) \wedge \mu(y) \wedge \nu(xy)$ (Since ν is an ideal of μ)

$\geq \eta(x) \wedge \mu(y) \wedge \nu(x) \wedge \mu(y)$ (Since ν is an ideal of μ)

$= \eta(x) \wedge \mu(y)$ (Since $\eta \subseteq \nu$)

Similarly $\eta(xy) \geq \eta(y) \wedge \mu(y)$. Thus η is an ideal of μ.■

Corollary 2.33. *Let R be a commutative ring and* $L(\mu, R)$ *be an L-ring. Let* η *be a prime ideal of* ν *and* ν *be an ideal of* μ. *Then* η *is an ideal of* μ.

Proof. Obvious.

Definition 2.34. Let $L(\mu, R)$ be an L-ring. An ideal η of μ is said to be *irreducible* if, whenever $\eta = \nu \cap \theta$, for some ideals ν and θ of μ, then either $\nu = \eta$ or $\theta = \eta$. An ideal η of μ is said to be *reducible* if it is not irreducible.

Definition 2.35. Let $L(\mu, R)$ be an L-ring and η be an ideal of μ. Let $\{(\eta_t, \mu_t)\}$ be the family of distinct pairs of non-empty level subsets. Then η is said to be a *prime ideal of* μ *of rank r* if there exists exactly r distinct pairs of level subsets, (η_t, μ_t) such that η_t is a prime ideal of μ_t and $\eta_t = \mu_t$ for all other pairs.

Clearly every prime ideal of μ of rank r is a prime ideal of μ.

Definition 2.36. Let $L(\mu, R)$ be an L-ring. An ideal η of μ is said to be *weakly prime* ideal of μ if for every pair of non-empty level subsets, (η_t, μ_t) with $\eta_t \neq R$, η_t is a prime ideal of μ_t.

Example 2.37. Let R be the ring of integers and L be a chain of four elements $t_3 < t_2 < t_1 < t_0$. Define $\mu : R \rightarrow L$ as follows :

$$\mu(x) = \begin{cases} t_0 & if\ x \in (2^3) \\ t_1 & if\ x \in (2^2) - (2^3) \\ t_2 & if\ x \in (2) - (2^2) \\ t_3 & if\ x \in R - (2)\ . \end{cases}$$

Then $L(\mu, R)$ is an L-ring. Define $\eta : R \to L$ as follows :

$$\eta(x) = \begin{cases} t_0 & \text{if } x \in (2^4) \\ t_1 & \text{if } x \in (2^3) - (2^4) \\ t_2 & \text{if } x \in (2^2) - (2^3) \\ t_3 & \text{if } x \in R - (2^2)\ . \end{cases}$$

Clearly η is a weakly prime ideal of μ. Define $\nu : R \to L$ as follows

$$\nu(x) = \begin{cases} t_0 & \text{if } x \in (2^4) \\ t_1 & \text{if } x \in (2^3) - (2^4) \\ t_2 & \text{if } x \in (2) - (2^3) \\ t_3 & \text{if } x \in R - (2)\ . \end{cases}$$

Clearly ν is a prime ideal of μ of rank 2. Moreover, ν is reducible since $\nu = \zeta \cap \theta$, where ζ and θ are ideals of μ defined by

$$\zeta(x) = \begin{cases} t_0 & \text{if } x \in (2^3) \\ t_2 & \text{if } x \in (2) - (2^3) \\ t_3 & \text{if } x \in R - (2) \end{cases}$$

$$\theta(x) = \begin{cases} t_0 & \text{if } x \in (2^4) \\ t_1 & \text{if } x \in (2^2) - (2^4) \\ t_2 & \text{if } x \in (2) - (2^2) \\ t_3 & \text{if } x \in R - (2)\ . \end{cases}$$

Theorem 2.38. *Let L be a chain and* $L(\mu, R)$ *be an L-ring. Let* η *be a prime ideal of rank 1 such that* η_{t_0} *is a prime ideal of* μ_{t_0} *for some* $t_0 \in \operatorname{Im}\mu$. *Then* η *is irreducible in* μ.

Proof. Let $\eta = \nu \cap \theta$, where ν and θ are ideals of μ. Then $\eta \subseteq \nu \subseteq \mu$ and $\eta \subseteq \theta \subseteq \mu$, and hence $\eta_t \subseteq \nu_t \subseteq \mu_t$ and $\eta_t \subseteq \theta_t \subseteq \mu_t$, $\forall\, t \in L$. Now $\eta_{t_0} = (\nu \cap \theta)_{t_0} = \nu_{t_0} \cap \theta_{t_0}$. By the hypothesis η_{t_0} is a prime ideal of μ_{t_0}. Since every prime ideal of a ring is irreducible, we have either $\nu_{t_0} = \eta_{t_0}$ or $\theta_{t_0} = \eta_{t_0}$. Before we show that η is irreducible, we prove that

$$\nu_t = \eta_t \text{ and } \theta_t = \eta_t, \quad \forall\, t \in L \text{ with } t_0 < t.$$

Let $t \in L$ with $t_0 < t$. Since $t_0 \in \operatorname{Im}\mu$, $\mu_t \subsetneq \mu_{t_0}$. Thus $(\eta_t, \mu_t) \neq (\eta_{t_0}, \mu_{t_0})$. Since η is a prime ideal of μ of rank 1 and η_{t_0} is a prime ideal of μ_{t_0}, we have $\eta_t = \mu_t$. By using $\eta_t \subseteq \nu_t \subseteq \mu_t$, we have $\nu_t = \eta_t$. Similarly $\theta_t = \eta_t$. In order to show that either $\nu = \eta$ or $\theta = \eta$, we consider the following cases.

Case (i) $\nu_{t_0} = \eta_{t_0}$ and $\theta_{t_0} \neq \eta_{t_0}$. As we have $\eta_t \subseteq \theta_t \subseteq \mu_t$, $\forall\, t \in L$, therefore $\eta_{t_0} \subsetneq \theta_{t_0}$. We prove that $\nu_t = \eta_t$, $\forall\, t \in \operatorname{Im}\nu$. If possible, there exists $t_1 \in \operatorname{Im}\nu$ such that $\nu_{t_1} \neq \eta_{t_1}$. As we have $\eta_t \subseteq \nu_t \subseteq \mu_t$, $\forall\, t \in L$, therefore $\eta_{t_1} \subsetneq \nu_{t_1}$. Since $\nu_t = \eta_t$, $\forall t \in L$ with $t_0 < t$, we have $t_1 < t_0$. Thus $\theta_{t_0} \subseteq \theta_{t_1}$. Since η is a prime ideal of μ and $\eta_{t_1} \subsetneq \nu_{t_1} \subseteq \mu_{t_1}$, by Theorem 2.2, η_{t_1} is a prime ideal of μ_{t_1}. Since η is a prime ideal of μ of rank 1 such that η_{t_0} is prime ideal of μ_{t_0}, we have $\eta_{t_1} = \eta_{t_0}$, $\mu_{t_1} = \mu_{t_0}$. Since every prime ideal of a ring is irreducible, η_{t_1} is an irreducible ideal of μ_{t_1}. Now $\eta_{t_1} = \nu_{t_1} \cap \theta_{t_1}$ and $\eta_{t_1} \subsetneq \nu_{t_1}$. Therefore $\eta_{t_1} = \theta_{t_1}$. Thus $\theta_{t_1} = \eta_{t_1} = \eta_{t_0} \subsetneq \theta_{t_0} \subseteq \theta_{t_1}$, which is a contradiction. Thus $\nu_t = \eta_t$, $\forall t \in \mathrm{Im}\,\nu$ and $\eta \subseteq \nu$. By Lemma 1.7, we have $\nu = \eta$.

Case (ii) $\theta_{t_0} = \eta_{t_0}$ and $\nu_{t_0} \neq \eta_{t_0}$. This case is similar to (i). In this case $\theta = \eta$.

Case (iii) $\nu_{t_0} = \eta_{t_0}$ and $\theta_{t_0} = \eta_{t_0}$. We show that either

$$\nu_t = \eta_t, \forall t \in \mathrm{Im}\,\nu \text{ or } \theta_t = \eta_t, \forall t \in \mathrm{Im}\,\theta.$$

If possible, there exists $t_1 \in \mathrm{Im}\,\nu$ and $t_2 \in \mathrm{Im}\,\theta$ such that $\nu_{t_1} \neq \eta_{t_1}$ and $\theta_{t_2} \neq \eta_{t_2}$. In view of the fact that $\eta_t \subseteq \nu_t \subseteq \mu_t$ and $\eta_t \subseteq \theta_t \subseteq \mu_t$, $\forall t \in L$, we have $\eta_{t_1} \subsetneqq \nu_{t_1}$ and $\eta_{t_2} \subsetneqq \theta_{t_2}$. We have shown above that $\nu_t = \eta_t$ and $\theta_t = \eta_t, \forall t \in L$ with $t_0 < t$. Therefore, we have $t_1 < t_0$ and $t_2 < t_0$. Since η is a prime ideal of μ and $\eta_{t_1} \subsetneqq \nu_{t_1} \subseteq \mu_{t_1}$, by Theorem 2.2, η_{t_1} is a prime ideal of μ_{t_1}. Moreover, η is a prime ideal of rank 1 such that η_{t_0} is a prime ideal of μ_{t_0}, and thus $\eta_{t_1} = \eta_{t_0}$, $\mu_{t_1} = \mu_{t_0}$. Now, since every prime ideal of a ring is irreducible, η_{t_1} is irreducible ideal of μ_{t_1}. Thus from $\eta_{t_1} = \nu_{t_1} \cap \theta_{t_1}$ and $\eta_{t_1} \subsetneqq \nu_{t_1}$, we have that $\eta_{t_1} = \theta_{t_1}$. Hence $\theta_{t_1} = \eta_{t_1} = \eta_{t_0} = \theta_{t_0} \subsetneqq \theta_{t_2}$, since $t_2 \in \mathrm{Im}\,\theta$ and $t_2 < t_0$. Thus $t_2 < t_1$. Similarly, we can show that $t_1 < t_2$, which is a contradiction. Thus we have either $\nu_t = \eta_t$, $\forall t \in \mathrm{Im}\,\nu$ or $\theta_t = \eta_t$, $\forall t \in \mathrm{Im}\,\theta$. We have either $\nu = \eta$ or $\theta = \eta$. Thus η is irreducible in μ.■

Theorem 2.39. *Let L be a complete chain and R be a commutative ring with unity. Let* $L(\mu, R)$ *be an L-ring and* η *be an ideal of* μ *and has sup property . Let* $\sqrt{\eta}$ *be a maximal ideal of* μ *such that* $(\sqrt{\eta})_{t_0}$ *is a maximal ideal of* μ_{t_0}, $t_0 \in \mathrm{Im}\,\mu$ *and* $\mu_{t_0} = R$ *. Then* η *is a primary ideal of* μ.

Proof. Let η_t be a non-empty level subset of μ_t such that $\eta_t \subsetneq \mu_t$. We show that η_t is primary ideal of μ_t. Now, two cases arise :

Case (i) $\left(\sqrt{\eta}\right)_t = \mu_t$. Then by Lemma 2.9, we have $\sqrt{\eta_t} \cap \mu_t = \mu_t$. Let $ab \in \eta_t$, $a, b \in \mu_t$ and $a \notin \eta_t$. Now $b \in \mu_t = \sqrt{\eta_t} \cap \mu_t \subseteq \sqrt{\eta_t}$. Thus η_t is a primary ideal of μ_t.

Case (ii) $\left(\sqrt{\eta}\right)_t \neq \mu_t$. By Theorem 1.12, we have $\left(\sqrt{\eta}\right)_t = \left(\sqrt{\eta}\right)_{t_0}$ and $\mu_t = \mu_{t_0} = R$. Therefore, we have

$$\left(\sqrt{\eta}\right)_{t_0} = \left(\sqrt{\eta}\right)_t = \sqrt{\eta_t} \cap \mu_t = \sqrt{\eta_t} \cap R = \sqrt{\eta_t}.$$

By the hypothesis, $\left(\sqrt{\eta}\right)_{t_0}$ is a maximal ideal of μ_{t_0}, therefore $\sqrt{\eta_t}$ is a maximal ideal of R. Consequently in view of a result of classical ring theory η_t is a primary ideal of R. That is, η_t is a primary ideal of μ_t.■

Theorem 2.40. *Let* $L(\mu, R)$ *be an L-ring. Let* η *be a prime ideal of* μ *and* ν *be an ideal of* μ*. Then* $\eta \cap \nu$ *is a prime ideal of* ν*.*

Proof. It is easy to verify that $\eta \cap \nu$ is an ideal of ν. To show that $\eta \cap \nu$ is a prime ideal of ν, let $x, y \in R$. Now

$$(\eta \cap \nu)(xy) \wedge \nu(x) \wedge \nu(y) = \eta(xy) \wedge \nu(xy) \wedge \nu(x) \wedge \nu(y)$$

$$= \eta(xy) \wedge \nu(x) \wedge \nu(y) \qquad (\nu(xy) \geq \nu(x) \wedge \nu(y))$$

$$= \eta(xy) \wedge \mu(x) \wedge \mu(y) \wedge \nu(x) \wedge \nu(y) \qquad (\mu(x) \geq \nu(x))$$

Since η is prime ideal of μ, we have either

$$\eta(xy) \wedge \mu(x) \wedge \mu(y) = \eta(x) \wedge \mu(y) \text{ or } \eta(xy) \wedge \mu(x) \wedge \mu(y) = \eta(y) \wedge \mu(x).$$

If $\eta(xy) \wedge \mu(x) \wedge \mu(y) = \eta(x) \wedge \mu(y)$, then we have

$$(\eta \cap \nu)(xy) \wedge \nu(x) \wedge \nu(y) = \eta(x) \wedge \mu(y) \wedge \nu(x) \wedge \nu(y)$$

$$= (\eta \cap \nu)(x) \wedge \nu(y)$$

Similarly, if $\eta(xy) \wedge \mu(x) \wedge \mu(y) = \eta(y) \wedge \mu(x)$, then we have

$$(\eta \cap \nu)(xy) \wedge \nu(x) \wedge \nu(y) = (\eta \cap \nu)(y) \wedge \nu(x).$$

Thus $\eta \cap \nu$ is a prime ideal of μ.■

Theorem 2.41. *Let L be a complete lattice and* $L(\mu, R)$ *be an L-ring. Let* ν *be a semiprime ideal of* μ*. Then for any ideal* η *of* μ *with* $\eta \subseteq \nu$*, we have* $\sqrt{\eta} \subseteq \nu$ *.*

Proof. Since ν is a semiprime ideal of μ, by Theorem 2.8, we have $\sqrt{\nu} = \nu$. Since $\eta \subseteq \nu$, by Theorem 2.12, we have $\sqrt{\eta} \subseteq \sqrt{\nu} = \nu$.■

Corollary 2.42. *Let L be a complete lattice and* $L(\mu, R)$ *be an L-ring. Let* η *be an ideal of* μ*. Then* η *and* $\sqrt{\eta}$ *are contained in the same prime ideal of* μ*.*

Proof. Obvious.

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