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RESIDUAL OF IDEALS OF AN L-RING

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ABSTRACT

In this paper we improve certain existing results in the theory of L-subrings. In particular, we present some characterizations of ideals and left ideals of an L-subring. Moreover, we introduce the concept of maximal ideal in an L-ring L(μ,R). Our approach is similar to that of classical ring theory. We also extend a well known result for maximal ideal of classical ring theory. Then we provide two different necessary conditions for an ideal η of an L-ring L(μ, R) to be maximal. We show by examples that these conditions are not sufficient. A partial converse of one of these conditions is also established.

INTRODUCTION

The concept of maximality of a subalgebra in a given algebra has been a topic of serious discussion and investigation in the studies of classical algebra. However, the same is not true in the field of fuzzy algebraic structures. The main reason behind this lack of attention towards this concept is the fact that in the fuzzy setting the subalgebras are the generalizations of subobjects of an object of a given category of algebra such as fuzzy subgroups of a group, fuzzy subrings and fuzzy ideals of a ring. In order to make such studies meaningful and more fruitful in the class of rings, we consider an L-subring μ of an ordinary ring R and we call the system L(μ,R) an L-ring. This system $L(\mu, R)$ can be considered as an object of a suitably defined category. Then we discuss the concept of subobjects of these L-rings in the form of L-ideals and L-left ideals of an L-subring μ.

It is shown that for a lattice L and a ring R, $\eta \in L^R$ is an L-ideal of L-subring μ if and only if level subset η_a is an ideal of level subset μ_a for all $a \in \mathrm{Im}\,\eta \cup \{ b \in L \mid b \le \eta(0) \}$. It is shown that for a chain L and a ring R, if $\eta \in L^R$ is an L-left ideal of μ, then the strong level subset $\eta_a^{>}$ is a left ideal of strong level subset $\mu_a^{>}$ for all $a \in L - \{1\}$. The converse of the above result is true provided L is a complete dense lattice. We have shown that the converse of the above result remains valid provided L is a chain. We have also shown that for a chain L, a subring η of an L-ring L(μ,R) is a left ideal (ideal) of μ if and only if η_a is a left ideal (ideal) of μ_a for all $a \in (\mathrm{Im}\eta \cup \mathrm{Im}\,\mu) \cap \{b \in L \mid b \leq \eta(0)\}$.

We extend the notion of maximality of an ideal of a ring to the L-setting. Our approach is similar to that of classical ring theory. For an L-ring L(μ,R), we say that a proper ideal η of μ is maximal in μ, if η is not properly contained in any proper ideal of μ. To justify our definition, we extend the following result of classical ring theory – An ideal I of a ring R is maximal in R if and only if for every $x \in R$ such that $x \notin I$, the ideal generated by $I \cup \{x\}$ is R. We prove that, for a chain L and a ring R, if an ideal η of an L-ring L(μ,R) is maximal then among all the distinct pairs of level subsets $\{(\eta_t, \mu_t)\}$, there is exactly one pair (η_{t_0}, μ_{t_0}) such that $\eta_{t_0} \subsetneq \mu_{t_0}$ and $\eta_t = \mu_t$ for all other pairs. Further, if a maximal ideal η of μ is such that $\eta(0) = \mu(0)$, then among all the distinct pairs of level subsets $\{(\eta_t, \mu_t)\}$, there is exactly one pair (η_{t_0}, μ_{t_0}) , such that η_{t_0} is a maximal ideal of μ_{t_0} and for all other pairs (η_t, μ_t), $\eta_t = \mu_t$. We also discuss the maximality of an ideal η in L(μ,R) in terms of the range set of μ. We also show that for a chain L and a ring R, if η is a maximal ideal of μ with η(0) = μ(0), then there exists $t_0 \in \mathrm{Im}\,\mu$ such that η_{t_0} is a maximal ideal of μ_{t_0} and $\eta_t = \mu_t$ for all $t \in \mathrm{Im}\,\mu - \{t_0\}$. We provide an example to show that the converse of this result is not valid. However, a partial converse of the above result is established which states that for a chain L and a ring R, and an ideal η of L-ring L(μ,R) if there exists $t_0 \in \mathrm{Im}\,\mu$ such that η_{t_0} is a maximal ideal of μ_{t_0} , $\eta_t = \mu_t$ for all $t \in \mathrm{Im}\mu - \{t_0\}$ and t_0 is cover of t_1 for some $t_1 \in \mathrm{Im}\mu$, then η is a maximal ideal of μ.

MAXIMAL IDEALS OF L-SUBRINGS – II

Definition 1.1. Let L be a lattice and R be a ring. Let $\mu \in L^R$. Then μ is called an *L-subring* of R if

(1) $\mu(x-y) \geq \mu(x) \wedge \mu(y)$ $\forall\, x, y \in R$, and

(2) $\mu(xy) \geq \mu(x) \wedge \mu(y)$ $\forall\, x, y \in R$.

The set of all L-subrings of R is denoted by L(R). It is obvious that if μ is an L-subring of R, then $\mu(x) \leq \mu(0)$, $\forall\ x \in R$. We shall call $\mu(0)$ to be the tip of the L-subring μ. For convenience, we use the notation $L(\mu, R)$ for the L-subring μ of R and we shall refer to it here as an L-ring $L(\mu, R)$. This terminology is adopted in view of the fact that in this paper, we shall mainly discuss the substructures of an L-ring $L(\mu,R)$ instead of fuzzy substructures of an ordinary ring R. In the next section, we shall define an ideal of an L-ring $L(\mu,R)$ and we will investigate its maximality in $L(\mu,R)$. It is worthwhile to mention here that in the earlier works on this topic the maximality of fuzzy substructures such as fuzzy subgroups of a group and fuzzy ideals of a ring were discussed by some authors.

Definition 1.2 Let $\mu \in L^R$. Then μ is called *L-ideal* of R if

(1) $\mu(x - y) \geq \mu(x) \wedge \mu(y) \quad \forall\ x, y \in R$, and

(2) $\mu(xy) \geq \mu(x) \vee \mu(y) \quad \forall\ x, y \in R$.

We denote the set of all L-ideals of R by LI(R). It is obvious that if R has identity 1 and $\mu \in LI(R)$, then $\mu(x) \geq \mu(1)$.

Definition 1.3 Let $\mu \in L^R$. Let

$$\langle \mu \rangle = \bigcap \{ \nu \mid \mu \subseteq \nu;\ \nu \in LI(R) \}$$

Then $\langle \mu \rangle$ is the smallest L-ideal of R containing μ and is called the *ideal generated* by μ.

Definition 1.4. Let $\mu \in L^X$. For $\alpha \in L$, we define *level subset* μ_α and *strong level subset* $\mu_\alpha^>$ of μ in X, as follows

$$\mu_\alpha = \{ x \in X \mid \mu(x) \geq \alpha \}$$

$$\mu_\alpha^> = \{ x \in X \mid \mu(x) > \alpha \}.$$

Obviously $\mu_\alpha^> \subseteq \mu_\alpha$ and for $\alpha \leq \beta$, $\mu_\beta \subseteq \mu_\alpha$ and $\mu_\beta^> \subseteq \mu_\alpha^>$.

Definition 1.5. Let $x \in X$ and $\alpha \in L$. We define $x_\alpha \in L^X$ as follows

$$x_\alpha(y) = \begin{cases} \alpha, & \text{if } y = x \\ 0, & \text{if } y \neq x \end{cases}$$

x_α is often referred to as an *L-point.*

PRELIMINARIES

Let X be a non-empty set and L be a lattice. By an L-subset of X, we mean a function from X to L. The set of all L-subsets of X is called the L-power set of X and is denoted by L^X. For $\mu \in L^X$, the set $\{\mu(x) | x \in X\}$ is called the image of μ, and is denoted by Imμ. For $\mu, \nu \in L^X$, if $\nu(x) \leq \mu(x)$, $\forall x \in X$, then we say that ν is contained in μ and we write $\nu \subseteq \mu$. If $\nu \subseteq \mu$ and $\nu \neq \mu$, then ν is said to be properly contained in μ and we write $\nu \underset{\neq}{\subset} \mu$. Throughout the paper, R will denote an ordinary ring.

Definition 1.1. Let L be a lattice and R be a ring. Let $\mu \in L^R$. Then μ is called an *L-subring* of R if

(1) $\mu(x - y) \geq \mu(x) \wedge \mu(y)$, $\forall\ x, y \in R$, and

(2) $\mu(xy) \geq \mu(x) \wedge \mu(y)$, $\forall\ x, y \in R$.

The set of all L-subrings of R is denoted by L(R). It is obvious that if μ is an L-subring of R, then $\mu(x) \leq \mu(0)$, $\forall\ x \in R$. We shall call μ(0) to be the tip of the L-subring μ. For convenience, we use the notation $L(\mu, R)$ for the L-subring μ of R and we shall refer to it here as an L-ring $L(\mu, R)$.

Definition 1.2 Let $\mu \in L^R$. Then μ is called *L-ideal* of R if

(1) $\mu(x - y) \geq \mu(x) \wedge \mu(y)$, $\forall\ x, y \in R$, and

(2) $\mu(xy) \geq \mu(x) \vee \mu(y)$, $\forall\ x, y \in R$.

We denote the set of all L-ideals of R by LI(R). It is obvious that if R has identity 1 and $\mu \in LI(R)$, then $\mu(x) \geq \mu(1)$.

Definition 1.3. Let $\mu \in L^X$. For $\alpha \in L$, we define *level subset* μ_α and *strong level subset* $\mu^>_\alpha$ of μ in X, as follows

$$\mu_\alpha = \{x \in X \mid \mu(x) \geq \alpha\}$$

$$\mu^>_\alpha = \{x \in X \mid \mu(x) > \alpha\}.$$

Obviously $\mu^>_\alpha \subseteq \mu_\alpha$ and for $\alpha \leq \beta$, $\mu_\beta \subseteq \mu_\alpha$ and $\mu^>_\beta \subseteq \mu^>_\alpha$.

Definition 1.4. Let $\nu \in L^R$ and $\mu \in L(R)$ with $\nu \subseteq \mu$. Then ν is called an *L-ideal* of μ (or in μ) if

(1) $\nu(x-y) \geq \nu(x) \wedge \nu(y)$, $\forall\ x, y \in R$, and

(2) $\nu(xy) \geq (\mu(x) \wedge \nu(y)) \vee (\nu(x) \wedge \mu(y))$, $\forall\ x, y \in R$

For convenience, ν is called an *ideal* of μ (or L-ring L(μ,R)).

Theorem 1.5 . *Let* $\eta \in L^R$ *and* $\mu \in L(R)$. *Then* η *is an L-ideal of* μ *if and only if,* $\forall$ $a \in \operatorname{Im}\eta \cup \{b \in L \mid b \leq \eta(0)\}$, η_a *is an ideal of* μ_a.

Theorem1.6. *Let L be a lattice and R be a ring. Let L(μ,R) be an L-ring and* $\eta \in L^R$ *with* $\eta \subseteq \mu$. *Then* η *is an ideal of* μ *if and only if each non-empty level subset* η_a *is an ideal of level subset* μ_a.

Lemma 1.7. Suppose L is a chain and R is a ring. Suppose $\theta, \mu \in L^R$ such that $\theta \subseteq \mu$ and $\theta_t = \mu_t$, $\forall$ $t \in \operatorname{Im}\mu$. Then $\theta = \mu$.

RESIDUAL OF IDEALS

Definition 2.1 . Let L be a complete lattice and $L(\mu, R)$ be an L-ring. Let η and ν be ideals of μ. The right quotient (residual) of η by ν , denoted by $[\eta :_r \nu]$, is defined by

$$[\eta :_r \nu] = \bigcup \{\xi \mid \xi \lhd \mu \text{ and } \xi o \nu \subseteq \eta\}.$$

The left quotient of η by ν, denoted by $[\eta :_l \nu]$, is define by

$$[\eta :_l \nu] = \bigcup \{\xi \mid \xi \lhd \mu \text{ and } \nu o \xi \subseteq \eta\}.$$

If R is commutative, then $[\eta :_r \nu] = [\eta :_l \nu]$. In this case it is called quotient of η by ν and is denoted by $[\eta : \nu]$.

Theorem 2.2. Let L be a complete Heyting algebra and $L(\mu, R)$ be an L-ring .Let η,ν be ideals of μ. Then $[\eta:_r \nu]$ and $[\eta:_l \nu]$ are ideals of μ. Also $\eta \subseteq [\eta:_r \nu] \subseteq \mu$ and $\eta \subseteq [\eta:_l \nu] \subseteq \mu$.

Proof. Let $x, y \in R$. Clearly $[\eta:_r \nu](-x) = [\eta:_r \nu](x)$. Write $A = \{\xi | \xi \lhd \mu \text{ and } \xi o \nu \subseteq \eta\}$. Suppose $\xi, \xi' \in A$. Then ξ and ξ' are ideals of μ such that $\xi o \nu \subseteq \eta$ and $\xi' o \nu \subseteq \eta$. By Theorem 1.12 $\xi + \xi'$ is an ideal of μ. Now by Lemma 1.7 and Lemma 1.8, we have

$$(\xi + \xi') o \nu \subseteq \xi o \nu + \xi' o \nu \subseteq \eta + \eta = \eta.$$

Thus $\xi + \xi' \in A$. Now

$$[\eta:_r \nu](x) \wedge [\eta:_r \nu](y) = \left[\bigvee_{\xi \in A} \xi(x)\right] \wedge \left[\bigvee_{\xi' \in A} \xi'(y)\right]$$

$$= \vee\{\xi(x) \wedge \xi'(y) \mid \xi, \xi' \in A\}$$

(Since L is complete Heyting algebra)

$$\leq \vee\{(\xi + \xi')(x + y) \mid \xi, \xi' \in A\}$$

$$\leq [\eta:_r \nu](x + y) \qquad \text{(Since } \xi + \xi' \in A\text{)}$$

Now $$[\eta:_r \nu](xy) = \left(\bigcup_{\xi \in A} \xi\right)(xy)$$

$$= \bigvee_{\xi \in A} \xi(xy)$$

$$\geq \bigvee_{\xi \in A} [\xi(x) \wedge \mu(y)]$$

$$= \left(\bigvee_{\xi \in A} \xi(x)\right) \wedge \mu(y) \qquad \text{(Since L is complete Heyting algebra)}$$

$$= [\eta:_r \nu](x) \wedge \mu(y)$$

Similarly $[\eta:_r \nu](xy) \geq [\eta:_r \nu](y) \wedge \mu(x)$. Thus $[\eta:_r \nu]$ is an ideal of μ. Clearly $[\eta:_r \nu] \subseteq \mu$. Since η is an ideal of μ, by Theorem 1.10, $\eta o \mu \subseteq \eta$. Thus by Lemma 1.7, we have $\eta o\nu \subseteq \eta o\mu \subseteq \eta$. Hence $\eta \in A$. Therefore $\eta \subseteq [\eta :_r \nu]$. Similarly $[\eta :_l \nu]$ is an ideal of μ and $\eta \subseteq [\eta :_l \nu] \subseteq \mu$.

Theorem 2.3. Let L be a complete Heyting algebra. Let $L(\mu, R)$ be an L-ring and η, ν be ideals of μ. Then

i) $[\eta :_r \nu]$ is the largest ideal of μ with the property that $[\eta :_r \nu]\nu \subseteq \eta$.

(ii) $[\eta :_l \nu]$ is the largest ideal of μ with the property that $\nu[\eta :_l \nu] \subseteq \eta$.

Proof. Write $A = \{\xi \mid \xi \triangleleft \mu \text{ and } \xi o\nu \subseteq \eta\}$. Then $[\eta :_r \nu] = \bigcup_{\xi \in A} \xi$. Let $x \in R$ and $x = \sum_{i=1}^{m} u_i w_i$. Now

$$\eta(u_i w_i) \geq (\xi o\nu)(u_i w_i) \geq \xi(u_i) \wedge \nu(w_i), \qquad \forall \xi \in A$$

Thus

$$\eta(u_i w_i) \geq \bigvee_{\xi \in A} [\xi(u_i) \wedge \nu(w_i)]$$

$$= \left[\bigvee_{\xi \in A} \xi(u_i)\right] \wedge \nu(w_i) \qquad \text{(Since L is a complete Heyting algebra)}$$

$$= [\eta :_r \nu](u_i) \wedge \nu(w_i)$$

Hence

$$\eta(x) = \eta\left(\sum_{i=1}^{m} u_i w_i\right)$$

$$\geq \bigwedge_{i=1}^{m} \eta(u_i w_i)$$

$$\geq \bigwedge_{i=1}^{m} \{[\eta :_r \nu](u_i) \wedge \nu(w_i)\}$$

Consequently

$$\eta(x) \geq \vee \left\{ \bigwedge_{i=1}^{m} [\eta :_r \nu](u_i) \wedge \nu(w_i) \mid x = \sum_{i=1}^{m} u_i w_i \right\}$$

$$= \left([\eta :_r \nu]\nu\right)(x)$$

Hence, $[\eta :_r \nu]\nu \subseteq \eta$.

Suppose λ is an ideal of μ such that $\lambda\nu \subseteq \eta$. By Lemma 1.7, $\lambda o \nu \subseteq \lambda\nu \subseteq \eta$. Thus $\lambda \in A$ and hence $\lambda \subseteq [\eta :_r \nu]$.

Theorem 2.4. Let L be a complete lattice and $L(\mu, R)$ be an L-ring. Let η, ν and θ be ideals of μ. Then the following assertions hold.

(i) If $\eta \subseteq \nu$, then $[\eta :_r \theta] \subseteq [\nu :_r \theta]$ and $[\theta :_r \nu] \subseteq [\theta :_r \eta]$,

(ii) If $\eta \subseteq \nu$ then $[\nu :_r \eta] = \mu$,

(iii) $[\eta :_r \eta] = \mu$,

(iv) If $\eta(0) = \nu(0)$, then $[\eta :_r \nu] = [\eta :_r \eta + \nu]$.

Proof. (i) Let $\eta \subseteq \nu$. Write $A = \{\xi | \xi \lhd \mu \text{ and } \xi o \theta \subseteq \eta\}$ and $B = \{\xi \mid \xi \lhd \mu \text{ and } \xi o \theta \subseteq \nu\}$.

Let $\xi \in A$. Then $\xi \lhd \mu$ and $\xi o \theta \subseteq \eta \subseteq \nu$. Thus $\xi \in B$. Hence $A \subseteq B$. Therefore

$$[\eta :_r \theta] = \bigcup_{\xi \in A} \xi \subseteq \bigcup_{\xi \in B} \xi = [\nu :_r \theta].$$

Similarly, we can show that $[\theta :_r \nu] \subseteq [\theta :_r \eta]$.

(ii) Let $\eta \subseteq \nu$. Write $A = \{\xi \mid \xi \lhd \mu \text{ and } \xi o \eta \subseteq \nu\}$. Now $\mu \lhd \mu$. Since $\eta \lhd \mu$, we have $\mu o \eta \subseteq \eta \subseteq \nu$. Thus $\mu \in A$ and hence

$$\mu \subseteq \bigcup_{\xi \in A} \xi = [\nu :_r \eta] \subseteq \mu.$$

Therefore $[\nu :_r \eta] = \mu$.

(iii) Obvious.

(v) Since $\eta(0)=\nu(0)$, by Theorem1.12, $\eta+\nu$ is an ideal of μ and $\nu\subseteq\eta+\nu$. By (i), $[\eta:_r\eta+\nu]\subseteq[\eta:_r\nu]$.

Write $A=\{\xi \mid \xi\lhd\mu \text{ and } \xi o\nu\subseteq\eta\}$ and $B=\{\xi\lhd\mu \text{ and } \xi o(\eta+\nu)\subseteq\eta\}$. Let $\xi\in A$ Then $\xi\lhd\mu$ and $\xi o\nu\subseteq\eta$. Since $\eta\lhd\mu$, $\xi o\eta\subseteq\mu o\eta\subseteq\eta$. Therefore, Lemma 1.7 and Lemma 1.8 , we have ,

$$\xi o(\eta+\nu)\subseteq\xi o\eta+\xi o\nu\subseteq\eta+\eta=\eta .$$

Hence $\xi\in B$. Thus

$$[\eta:_r\nu]=\bigcup_{\xi\in A}\xi\subseteq\bigcup_{\xi\in B}\xi=[\eta:_r\eta+\nu].$$

Consequently

$$[\eta:_r\eta+\nu]=[\eta:_r\nu].$$

Similar results hold for left quotients.

Corollary 2.5. Let L be a complete Heyting algebra and $L(\mu,R)$ be an L-ring. Let η and ν be ideals of μ Then

(i) $\big[[\eta:_r\nu]:_r\eta\big]=\mu$,

(ii) $[\eta:_r\eta\nu]=\mu$,

(iii) $\big[[\eta:_r\eta]_r:\nu\big]=\mu$,

(iv) $[\eta:_r(\eta\cap\nu)]=\mu$,

(v) $[(\eta\cap\nu):_r\eta\nu]=\mu$.

Proof. (i) Since $\eta\subseteq[\eta:_r\nu]$, by Theorem 2.4 (ii), we have $\big[[\eta:_r\nu]:_r\eta\big]=\mu$.

(ii) By Theorem1.13 , $\eta\nu$ is an ideal of μ. Since η is an ideal of μ, by Lemma 1.7 and Lemma 1.11, we have

$$\eta\nu\subseteq\eta\mu\subseteq\eta.$$

Therefore, by Theorem 2.4 (ii), we have $[\eta:_r\eta\nu]=\mu$.

(iii) By Theorem 2.4 (iii), $[\eta:_r\eta]=\mu$. Since $\nu\subseteq\mu=[\eta:_r\eta]$, by Theorem 2.4 (ii), we have $\big[[\eta:_r\eta]_r:\mu\big]=\mu$.

(iv) Since $\eta \cap \nu$ is an ideal of μ and $\eta \cap \nu \subseteq \eta$, by Theorem 2.4(ii), we have $[\eta :_r (\eta \cap \nu)] = \mu$.

(v) $\eta \cap \nu$ and $\eta\nu$ are ideals of μ. Since $\eta\nu \subseteq \eta$ and $\eta\nu \subseteq \nu$, we have $\eta\nu \subseteq \eta \cap \nu$ .Therefore by Theorem 2.4 (ii), we have $[(\eta \cap \nu) :_r \eta\nu] = \mu$.

Corollary 2.6. Let L be a complete Heyting algebra and $L(\mu, R)$ be L-ring. Let η and ν are ideals of μ with $\eta(0) = \nu(0)$. Then

(i) $[(\eta+\nu) :_r \eta] = \mu$,

(ii) $[(\eta+\nu) :_r (\eta \cap \nu)] = \mu$,

(iii) $[(\eta+\nu) :_r \eta\nu] = \mu$.

Proof. (i) By Theorem 1.12, $\eta + \nu$ is an ideal of μ and $\eta \subseteq \eta + \nu$. Thus by Theorem 2.4 (ii), we have $[(\eta + \nu) :_r \eta] = \mu$.

(ii) Since $(\eta \cap \nu) \subseteq (\eta + \nu)$, by Theorem 2.4 (ii), we have $[(\eta + \nu) :_r (\eta \cap \nu)] = \mu$.

(iii) Since $\eta\nu \subseteq \eta \subseteq \eta + \nu$, by Theorem 2.4 (ii), we have $[(\eta+\nu) :_r \eta\nu] = \mu$.

Theorem 2.7. Let L be a complete Heyting algebra. Let $\eta_1, \eta_2, \ldots, \eta_m$, ν and θ be ideals of μ, Then

(i) $\left[\bigcap_{i=1}^{m} \eta_i :_r \nu\right] = \bigcap_{i=1}^{m} [\eta_i :_r \nu]$.

(ii) $\left[\nu :_r \sum_{i=1}^{m} \eta_i\right] = \bigcap_{i=1}^{m} [\nu :_r \eta_i]$ (provided $\eta_i(0) = \eta_j(0)$ $\forall$ i, j).

Proof. (i) Since $\bigcap_{j=1}^{m} \eta_j \subseteq \eta_i$ $\forall$ i , by Theorem 2.4(i), we have

$$\left[\bigcap_{i=1}^{m} \eta_i :_r \nu\right] \subseteq [\eta_i :_r \nu], \quad \forall\, i .$$

Hence $\left[\bigcap_{i=1}^{m} \eta_i :_r \nu\right] \subseteq \bigcap_{i=1}^{m} [\eta_i :_r \nu]$.

Write $A = \{\xi \mid \xi \triangleleft \mu \text{ and } \xi o\nu \subseteq \eta_1\}$, $B = \{\xi \mid \alpha \triangleleft \mu \text{and } \xi o\nu \subseteq \eta_2\}$,and $C = \{\xi \mid \xi \triangleleft \mu \text{ and } \xi o\nu \subseteq \eta_1 \cap \eta_2\}$ .Let $x \in R$. Now

$$\left([\eta_1 :_r \nu] \cap [\eta_2 :_r \nu]\right)(x) = \left[\left(\bigcup_{\xi \in A} \xi\right) \cap \left(\bigcup_{\xi \in B} \xi'\right)\right](x)$$

$$= \left(\bigvee_{\xi \in A} \xi(x)\right) \wedge \left(\bigvee_{\xi' \in B} \xi'(x)\right)$$

$$= \bigvee_{\xi' \in B} \left[\left(\bigvee_{\xi \in A} \xi(x)\right) \wedge \xi'(x)\right]$$

(Since L is a complete Heyting algebra)

$$= \bigvee_{\xi' \in B} \left\{ \bigvee_{\xi \in A} \left(\xi(x) \wedge \xi'(x)\right)\right\}$$

(Since L is a complete Heyting algebra)

$$= \vee \{\xi(x) \wedge \xi'(x) \mid \xi \in A, \xi' \in B\}.$$

Let $\xi \in A$ and $\xi' \in B$. Then ξ and ξ' are ideals of μ. Also $\xi o\nu \subseteq \eta_1$ and $\xi' o\nu \subseteq \eta_2$. Now by Lemma 1.9, $\xi \cap \xi'$ is an ideal of μ and by Lemma 1.7, we have

$$(\xi \cap \xi')o\nu \subseteq (\xi o\nu) \cap (\xi' o\nu) \subseteq \eta_1 \cap \eta_2.$$

Thus $\xi \cap \xi' \in C$. Hence

$$[\eta_1 \cap \eta_2 :_r \nu] = \bigcup_{\xi \in C} \xi \supseteq \bigcup \{\xi \cap \xi' \mid \xi \in A, \xi' \in B\}.$$

Therefore

$$[\eta_1 \cap \eta_2 :_r \nu](x) \geq \vee\{(\xi \cap \xi')(x) \mid \xi \in A, \xi' \in B\}$$

$$= \vee\{\xi(x) \wedge \xi'(x) \mid \xi \in A, \xi' \in B\}$$

$$= \left([\eta_i :_r \nu] \cap [\eta_2 :_r \nu]\right)(x)$$

Hence

$$[\eta_1 :_r \nu] \cap [\eta_2 :_r \nu] \subseteq [\eta_1 \cap \eta_2 :_r \nu].$$

Consequently

$$[\eta_1 \cap \eta_2 :_r \nu] = [\eta_1 :_r \nu] \cap [\eta_2 :_r \nu].$$

(ii) By Theorem 1.12, $\eta_1 + \eta_2$ is an ideal of μ such that $\eta_1 \subseteq \eta_1 + \eta_2$ and $\eta_2 \subseteq \eta_1 + \eta_2$. Thus by Theorem 2.4(i), we have

$$[\nu :_r \eta_1 + \eta_2] \subseteq [\nu :_r \eta_1] \text{ and } [\nu :_r \eta_1 + \eta_2] \subseteq [\nu :_r \eta_2]$$

Hence $[\nu :_r \eta_1 + \eta_2] \subseteq [\nu :_r \eta_1] \cap [\nu :_r \eta_2]$.

Write $A = \{\xi \mid \xi \lhd \mu \text{ and } \xi o \eta_1 \subseteq \nu\}$, $B = \{\xi \mid \xi \lhd \mu \text{ and } \xi o \eta_2 \subseteq \nu\}$ and $C = \{\xi \mid \xi \lhd \mu \text{ and } \xi o(\eta_1 + \eta_2) \subseteq \nu\}$.

Let $x \in R$. Then $\left([\nu :_r \eta_1 + \eta_2]\right)(x) = \underset{\xi \in C}{\vee} \xi(x)$. Now

$$\left([\nu :_r \eta_1] \cap [\nu :_r \eta_2]\right)(x) = [\nu :_r \eta_1](x) \wedge [\nu :_r \eta_2](x)$$

$$= \left(\underset{\xi \in A}{\vee} \xi(x)\right) \wedge \left(\underset{\xi \in B}{\vee} \xi'(x)\right)$$

$$= \vee\{\xi(x) \wedge \xi'(x) \mid \xi \in A, \xi' \in B\}$$

(Since L is a complete Heyting algebra)

Let $\xi \in A$, $\xi' \in B$. Then ξ and ξ' are ideals of μ. Also $\xi o \eta_1 \subseteq \nu$. $\xi' o \eta_1 \subseteq \nu$. Now by Lemma 1.9, $\xi \cap \xi'$ is an ideal of μ. Now

$$(\xi \cap \xi') o(\eta_1 + \eta_2) \subseteq (\xi \cap \xi') o \eta_1 + (\xi \cap \xi') o \eta_2 \qquad \text{(by Lemma 1.7)}$$

$$\subseteq \xi o \eta_1 + \xi' o \eta_2$$

$$\subseteq \nu + \nu = \nu.$$

Thus $\xi \cap \xi' \in C$. Hence

$$[\nu :_r (\eta_1 + \eta_2)] = \bigcup_{\lambda \in C} \lambda \supseteq \bigcup \{\xi \cap \xi' \mid \xi \in A, \xi' \in B\}.$$

Therefore

$$[\nu :_r \eta_1 + \eta_2](x) \geq \vee \{\xi(x) \wedge \xi'(x) \mid \xi \in A, \xi' \in B\}$$

$$=([\nu :_r \eta_1]\cap[\nu :_r \eta_2])(x)$$
$$=([\nu :_r \eta_1]\cap[\nu :_r \eta_2])(x)$$

Thus $[\nu :_r \eta_1+\eta_2]=[\nu :_r \eta_1]\cap[\nu :_r \eta_2]$.

Theorem 2.8. Let L be a complete Heyting algebra and $L(\mu, R)$ be an L-ring. Let η, ν and θ be ideals of μ. Then

(i) $[\eta :_r \nu\theta]=[[\eta :_r \theta] :_r \nu]$

(ii) $[\eta :_l \nu\theta]=[[\eta :_l \nu] :_l \theta]$.

Proof. By Theorem1.13, $\nu\theta$ is an ideal of μ. Write $A=\{\xi \mid \xi \lhd \mu \text{ and } \xi o \nu \subseteq [\eta :_r \theta]\}$ and $B=\{\xi \mid \xi \lhd \mu \text{ and } \xi o(\nu\theta) \subseteq \eta\}$. Then

$[[\eta :_r \theta] :_r \nu]=\cup\{\xi \mid \xi \in A\}$ and $[\eta :_r \nu\theta]=\cup[\xi \mid \xi \in B]$.

To show that $[\eta :_r \nu\theta]=[[\eta :_r \theta] :_r \nu]$, it is sufficient to prove that A =B.

Let $\xi \in A$. Then $\xi \lhd \mu$ and $\xi o \nu \subseteq [\eta :_r \theta]$. By Lemma 1.14, we have $\xi\nu \subseteq [\eta :_r \theta]$. By Theorem 2.3, $[\eta :_r \theta]\theta \subseteq \eta$. Thus by Theorem 1.7, we have

$$\xi(\nu\theta)=(\xi\nu)\theta \subseteq [\eta :_r \theta]\theta \subseteq \eta.$$

Hence by Lemma 1.14, we have $\xi o(\nu\theta) \subseteq \eta$. Therefore $\xi \in B$. Thus $A \subseteq B$. Conversely, suppose that $\xi \in B$. Then $\xi \lhd \mu$ and $\xi o(\nu\theta) \subseteq \eta$. By Lemma 1.14, we have $\xi(\nu\theta) \subseteq \eta$. Hence by Lemma 1.7, we have $(\xi\nu)\theta=\xi(\nu\theta) \subseteq \eta$. Since ξ and ν are ideal of μ, by Lemma 1.13, $\xi\nu$ is an ideal of μ. That is $\xi\nu$ is an ideal of μ and $(\xi\nu)\theta \subseteq \eta$. By Theorem 2.3, $[\eta :_r \theta]$ is the largest ideal of μ such that $[\eta :_r \theta]\theta \subseteq \eta$. Therefore $\xi\nu \subseteq [\eta :_r \theta]$. Hence by Lemma 1.14, we $\xi o \nu \subseteq [\eta :_r \theta]$. Thus $\xi \in A$. Therefore $B \subseteq A$. Consequently $A=B$.

Theorem 2.9. Let L be a complete Heyting algebra and $L(\mu, R)$ be an L-ring. Let η, ν and θ be ideals of μ. Then

(i) $[\eta :_r \nu] \subseteq [\eta\theta :_r \nu\theta]$

(ii) $[\eta :_l \nu] \subseteq [\theta\eta :_l \theta\nu]$.

Proof. By Theorem 1.13, $\eta\theta$ and $\nu\theta$ are ideals of μ. Write $A = \{\xi \mid \xi \lhd \mu \text{ and } \xi o \nu \subseteq \eta\}$ and $B = \{\xi \mid \xi \lhd \mu \text{ and } \xi o(\nu\theta) \subseteq \eta\theta\}$. Then

$$[\eta :_r \nu] = \cup \{\xi \mid \xi \in A\} \text{ and } [\eta\theta :_r \nu\theta] = \cup \{\xi \mid \xi \in B\}.$$

Suppose $\xi \in A$. Then $\xi \lhd \mu$ and $\xi o \nu \subseteq \eta$. Therefore by Lemma 1.14, we have $\xi\nu \subseteq \eta$. Thus by Lemma 1.7, we have

$$\xi(\nu\theta) = (\xi\,\nu)\theta \subseteq \eta\theta .$$

Hence, by Lemma 1.14, we have, $\xi o(\nu\theta) \subseteq \eta\theta$. Thus $\xi \in B$. Hence $A \subseteq B$. Therefore

$$[\eta :_r \nu] = \cup \{\xi \mid \xi \in A\} \subseteq \cup \{\xi \mid \xi \in B\} = [\eta\theta :_r \nu\theta].$$

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