International Journal of Analysis of Basic and Applied Science http://www.bharatpublication.com

(IJABAS) 2017, Vol. No. 1, Issue No. I, Oct-Dec **ISSN: 2457-0451**

THEOREMS RELATED TO IDEALS OF AN L-RING

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ABSTRACT

In this paper, we develop a systematic theory for the ideals of an L-ring $L(\mu, R)$. We introduce the concepts of prime ideal, semiprime ideal, primary ideal and radical of an ideal in an L-ring. We prove several results pertaining to these notions which are versions of their counter part in classical ring theory. Besides this we prove that for a commutative ring R, the radical $\sqrt{\eta}$ of a primary ideal η of an L-ring $L(\mu, R)$ is a prime ideal of μ provided η has sup-property. Moreover we introduce the concepts of minimal prime ideal and that of irreducibility of an ideal. Furthermore, we introduce the concept of semiprime radical of ideal in an L-ring. Among various results pertaining to this concept, we prove here that semiprime radicals of an ideal η, its radical $\sqrt{\eta}$, and its semiprime radical $S(\eta)$, all coincide.

INTRODUCTION

We introduced the concept of a maximal ideal of an L-ring $L(\mu, R)$. That is, we discussed the maximality of an ideal η in the L -subring μ of R . In this paper, we have introduced the concepts of prime ideal, semiprime ideal and primary ideal of an L -ring. These concepts provide a systematic development of the theory of ideals in an L-ring. The concept of the radical of an ideal in an L-ring is also introduced in this paper. The radical of an ideal of η of an L -ring $L(\mu, R)$ is denoted by $\sqrt{\eta}$. It is proved that an ideal η of μ is semiprime if and only if $\sqrt{\eta} = \eta$. It is also proved that for a commutative ring R, the radical $\sqrt{\eta}$ of an ideal η of an L -ring $L(\mu, R)$ is an ideal of μ . We have established some results pertaining to the notions of radical of an ideal of an L -ring which are versions of corresponding results of classical ring theory. It is proved that every semiprime ideal of an L -ring which is also primary is a prime ideal of the L -ring. It is also proved that if R is a commutative ring, ν is an ideal of L -ring $L(\mu, R)$ and η is a semiprime ideal of ν then η is an ideal of μ . We have also shown that for a commutative ring R, the radical $\sqrt{\eta}$ of a primary ideal of η of an $L-$ring $L(\mu, R)$ is a prime ideal of μ, provided η has sup property.

The concept of minimal prime ideal of an ideal of L-ring is introduced and its existence is established. It is proved that if an ideal η of an L-ring $L(\mu, R)$ is contained in some prime ideal of μ, then a minimal prime ideal of η exists. We have also introduced the concept of irreducibility of an ideal of an L-ring. We have shown that for a chain L, every prime ideal of rank 1 is irreducible under certain conditions. In classical ring theory, it is well known that if the radical $\sqrt{I}$ of an ideal I of a ring R is maximal, then I is primary ideal. We have established the corresponding result in an L-ring.

We have introduced the concept of semiprime radical of an ideal η of μ and which is denoted by $S(\eta)$. The semiprime radicals of an ideal η, its radical $\sqrt{\eta}$, and its semiprime radical $S(\eta)$ all coincide. It is also proved that the semiprime radical of an ideal of an L-ring $L(\mu, R)$ is the smallest semiprime ideal of μ containing the radical of the ideal. It is shown that for a commutative ring R and a complete Heyting algebra L, the radical and semiprime radical of an ideal of an L-ring $L(\mu, R)$ are identical.

MAXIMAL IDEALS OF AN L-RING

The definition of an ideal of an L-ring allows us to formulate the concept of maximal ideal of an L-ring L(μ,R) in the spirit of classical ring theory. Recall that a proper ideal I of an ordinary ring R is maximal in R if it is not properly contained in any other ideal of R.

Definition 2.1. Suppose L is a lattice and R is a ring. A proper ideal η of an L-ring L(μ,R) is said to be a *maximal ideal* of μ if for any ideal θ of μ, whenever $\eta \subseteq \theta \subseteq \mu$, then either $\theta = \eta$ or $\theta = \mu$.

The following result is straightforward.

Theorem 2.2. *Let L be a complete lattice and R be a ring. Let L(μ,R) be an L-ring. Then the intersection of an arbitrary family of ideals of μ is an ideal of μ.*

The above theorem ensures the existence of an ideal which is generated by an L-subset η contained in μ.

Definition 2.3. Let L be a complete lattice and R be a ring. Let L(μ,R) be an L-ring and $\eta \in L^R$ with $\eta \subseteq \mu$. Then the smallest ideal of μ containing η, that is,

$$\bigcap \{\nu \mid \eta \subseteq \nu \subseteq \mu, \nu \text{ is an ideal of } \mu\}$$

is called the *ideal of μ generated by η* and is denoted by $\langle \eta \rangle$.

The above concept provides us an extension of a well known result of classical ring theory.

Theorem 2.4. *Let L be a complete lattice and R be a ring. Let L(μ,R) be an L-ring. Then an ideal θ of μ is maximal in μ if and only if,* $\langle \theta, x_\alpha \rangle = \mu$ *for each L-point* x_α *of R satisfying* $\theta(x) < \alpha \leq \mu(x)$, *where* $\langle \theta, x_\alpha \rangle$ *is the ideal of μ generated by* $\theta \cup x_\alpha$.

Proof. Suppose θ is a maximal ideal of μ and x_α is an L-point of R such that $\theta(x) < \alpha \leq \mu(x)$. Now

$$(\theta \cup x_\alpha)(y) = \theta(y) \vee x_\alpha(y) = \begin{cases} \theta(y), & \text{if } y \neq x \\ \alpha, & \text{if } y = x \end{cases}.$$

Since $(\theta \cup x_\alpha)(x) = \alpha > \theta(x)$, θ is properly contained in the ideal $\langle \theta, x_\alpha \rangle$. Also $\langle \theta, x_\alpha \rangle \subseteq \mu$ and θ is maximal ideal of μ. Hence $\langle \theta, x_\alpha \rangle = \mu$. Conversely, suppose $\langle \theta, x_\alpha \rangle = \mu$ for all L-points x_α of R satisfying $\theta(x) < \alpha \leq \mu(x)$. Let η be an ideal of μ such that $\theta \underset{\neq}{\subset} \eta \subseteq \mu$. Then for some $a \in R$, $\theta(a) < \eta(a) \leq \mu(a)$. Write $\eta(a) = \alpha$. Now

$$(\theta \cup a_\alpha)(x) = \begin{cases} \theta(x) & , \quad if\ x \neq a \\ \alpha & , \quad if\ x = a \end{cases}$$

Thus $\theta \cup a_\alpha \subseteq \eta$. Hence $\mu = \langle \theta, a_\alpha \rangle \subseteq \langle \eta \rangle = \eta \subseteq \mu$. Therefore $\eta = \mu$. Hence θ is maximal ideal of μ. ν

Corollary 2.5. *Let L be a complete chain and R be a ring. Let L(μ,R) be an L-ring. Then an ideal θ of μ is maximal in μ if and only if* $\langle \theta, x_\alpha \rangle = \mu$ *for all L-points* x_α *of R contained in μ but* $x_\alpha \not\subseteq \theta$.

Our next result exhibits that for a general lattice L and an L-ring L(μ,R) , if an ideal η is maximal in μ then either the tips of η and μ are identical or the tip of μ covers the tip of η.

Theorem 2.6. *Let L be a lattice and R be a ring. Let L(μ,R) be an L-ring. If η is a maximal ideal of μ then there is no element* $t_0 \in L$ *such that* $\eta(0) < t_0 < \mu(0)$.

Proof. Suppose there exists an element $t_0 \in L$ such that $\eta(0) < t_0 < \mu(0)$. Define an L-subset $\theta : R \to L$ by

$$\theta(x) = \begin{cases} t_0, & x = 0 \\ \eta(x), & otherwise \end{cases}$$

Then $\eta \underset{\neq}{\subset} \theta \underset{\neq}{\subset} \mu$ and clearly in view of Theorem 2.15, θ is an ideal of μ, which contradicts the maximality of η. ν

Corollary 2.7. *Suppose L(μ,R) is an L-ring. If η is a maximal ideal of μ, then either* $\eta(0) = \mu(0)$ *or* $\mu(0)$ *is cover of η(0).*

Corollary 2.8. *Suppose L is a dense lattice and L(μ,R) is an L-ring. If η is a maximal ideal of μ, then η(0) = μ(0).*

Since [0,1] is a dense lattice, the Corollary 3.8, remains valid for ideals of a fuzzy ring (μ,R). From now onwards, L will always denote a chain, unless otherwise, specifically mentioned. Moreover, for an L-subset η contained in the L-subset μ, the set $\{(\eta_t, \mu_t)\}$ denotes the collection of all distinct pairs of level subsets of η and μ in R.

Theorem 2.9. *Suppose L(μ,R) is an L-ring and η is a maximal ideal of μ. Then there is exactly one pair* (η_{t_0}, μ_{t_0}) *such that* $\eta_{t_0} \underset{\neq}{\subset} \mu_{t_0}$ *and for all other pairs* (η_t, μ_t)*, we have* $\eta_t = \mu_t$.

Proof. Since η is a maximal ideal of μ, we have $\eta \underset{\neq}{\subset} \mu$. Thus $\eta_t \subseteq \mu_t$, $\forall$ t ∈L and there exists $t_0 \in R$ such that $\eta(x_0) < \mu(x_0) = t_0$ (say). Hence $\eta_{t_0} \underset{\neq}{\subset} \mu_{t_0}$. Suppose there exists $(\eta_{t_0}, \mu_{t_0}), (\eta_{t_1}, \mu_{t_1}) \in \{(\eta_t, \mu_t)\}$, such that $\eta_{t_0} \underset{\neq}{\subset} \mu_{t_0}$ and $\eta_{t_1} \underset{\neq}{\subset} \mu_{t_1}$ For the sake of definiteness, assume that $t_0 < t_1$.

Define an L-subset $\theta : R \to L$, as follows

$$\theta(x) = \begin{cases} t_0 & , \quad x \in \mu_{t_0} - \eta_{t_0} \\ \mu(x) & , \quad x \in R - \mu_{t_0} \\ \eta(x) & , \quad x \in \eta_{t_0} \end{cases}$$

We show that $\eta \underset{\neq}{\subset} \theta \underset{\neq}{\subset} \mu$. For $x \in \mu_{t^0} - \eta_{t^0}$, $\eta(x) < t_0 = \theta(x) \le \mu(x)$. For $x \in (R - \mu_{t_0}) \cup \eta_{t_0}$, obviously, $\eta(x) \le \theta(x) \le \mu(x)$. Thus $\eta \underset{\neq}{\subset} \theta \subseteq \mu$. Next, since $\eta_{t_1} \underset{\neq}{\subset} \mu_{t_1}$, the subset $\mu_{t_1} - \eta_{t_1}$ is non-empty. Let $x_0 \in \mu_{t_1} - \eta_{t_1}$. Then $x_0 \notin \eta_{t_1}$ and $x_0 \in \mu_{t_1} - \eta_{t_1} \subseteq \mu_{t_1} \subseteq \mu_{t_0}$. Now, either $x_0 \in \eta_{t_0}$ or $x_0 \notin \eta_{t_0}$. Now, if $x_0 \in \eta_{t_0}$, then since $x_0 \notin \eta_{t_1}$, we have $x_0 \in \eta_{t_0} - \eta_{t_1}$ and hence by the definition of θ, $\theta(x_0) = \eta(x_0) < t_1 \le \mu(x_0)$. And, if $x_0 \notin \eta_{t_0}$, then $x_0 \in \mu_{t_0} - \eta_{t_0}$. Therefore, $\theta(x_0) = t_0 < t_1 \le \mu(x_0)$ and thus $\eta \underset{\neq}{\subset} \theta \underset{\neq}{\subset} \mu$.

To show that θ is an ideal of μ, let θ_t be a non-empty level subset. We consider the following cases.

Case (i) $t = t_0$. We show that $\theta_{t_0} = \mu_{t_0}$. Since $\theta \subseteq \mu$, we have $\theta_{t_0} \subseteq \mu_{t_0}$. To show the reverse inclusion, let $x \in \mu_{t_0} = (\mu_{t_0} - \eta_{t_0}) \cup \eta_{t_0}$. Now, if $x \in \mu_{t_0} - \eta_{t_0}$, then $\theta(x) = t_0$. Therefore, $x \in \theta_{t_0}$ and hence $\mu_{t_0} - \eta_{t_0} \subseteq \theta_{t_0}$. Also $\eta_{t_0} \subseteq \theta_{t_0}$. Consequently, $\mu_{t_0} \subseteq \theta_{t_0}$. Now since the level subset μ_{t_0} is an ideal of itself, θ_{t_0} is an ideal of μ_{t_0}.

Case (ii) $t > t_0$. We show that $\theta_t = \eta_t$. Clearly $\eta_t \subseteq \theta_t$. Let $x_0 \in \theta_t$. Then $\theta(x_0) \ge t > t_0$. Now by the definition of θ, $x_0 \notin \mu_{t_0} - \eta_{t_0}$ and thus $x_0 \in (R - \mu_{t_0}) \cup \eta_{t_0}$. Suppose, if $x_0 \in R - \mu_{t_0}$ then $\theta(x_0) = \mu(x_0) < t_0$. This contradicts that $\theta(x) > t_0$. Thus we must have, $x_0 \in \eta_{t_0}$, and by the definition of θ, $\eta(x_0) = \theta(x_0) \ge t$. Therefore $x_0 \in \eta_t$ and hence $\theta_t \subseteq \eta_t$. Consequently $\theta_t = \eta_t$. Since η is an ideal of μ, the level subset η_t is an ideal of the level subset μ_t. That is, θ_t is an ideal of the level subset μ_t.

Case (iii) $t < t_0$. We show that $\theta_t = \mu_t$. In view of the fact that $\theta \subseteq \mu$, it is sufficient to show that $\mu_t \subseteq \theta_t$. Let $x_0 \notin \theta_t$. Then $\theta(x_0) < t < t_0$. Now by the definition of θ, we have $x_0 \notin \mu_{t_0} - \eta_{t_0}$. Therefore $x_0 \in \eta_{t_0} \cup (R - \mu_{t_0})$. Suppose, if $x_0 \in \eta_{t_0}$, then $\theta(x_0) = \eta(x_0) \ge t_0$, which is a contradiction. So that $x_0 \in R - \mu_{t_0}$. Again by the definition of θ, we have $\mu(x_0) = \theta(x_0) < t$ and hence $x_0 \notin \mu_t$. Thus $\mu_t \subseteq \theta_t$. Now, since the level subset μ_t is an ideal of itself, θ_t is an ideal of μ_t.

Therefore, each non-empty level subset θ_t is an ideal of the level subset μ_t and hence, θ is an ideal of μ. This contradicts the maximality of η in μ. Consequently, there is exactly one pair (η_{t_0}, μ_{t_0}) such that $\eta_{t_0} \underset{\neq}{\subset} \mu_{t_0}$ and for all other pairs (η_t, μ_t), we have $\eta_t = \mu_t$. ν

Now we prove our main theorem.

Theorem 2.10. *Suppose* $L(\mu, R)$ *is an L-ring and η is a maximal ideal of μ with η(0) = μ(0). Then there exists exactly one pair* (η_{t_0}, μ_{t_0}) *such that* η_{t_0} *is a maximal ideal of* μ_{t_0} *and for all other pairs* (η_t, μ_t), *we have* $\eta_t = \mu_t$.

Proof. By the above theorem, there exists exactly one pair (η_{t_0}, μ_{t_0}) such that $\eta_{t_0} \underset{\neq}{\subset} \mu_{t_0}$ and for all other pairs (η_t, μ_t), we have $\eta_t = \mu_t$. Since $\eta_{t_0} \underset{\neq}{\subset} \mu_{t_0}$, the level subset μ_{t_0} is non-empty. Hence $t_0 \leq \mu(0) = \eta(0)$ and thus η_{t_0} is non-empty. We show that η_{t_0} is a maximal ideal of μ_{t_0}. If possible, there exists an ideal I of μ_{t_0} such that $\eta_{t_0} \underset{\neq}{\subset} I \underset{\neq}{\subset} \mu_{t_0}$.

Firstly we prove that $I \underset{\neq}{\subset} \eta_t$, $\forall$ $t \in \operatorname{Im}\eta$ such that $t < t_0$. Suppose $t \in \operatorname{Im}\eta$ and $t < t_0$. Then $\eta_{t_0} \underset{\neq}{\subset} \eta_t$ and hence $(\eta_t, \mu_t) \neq (\eta_{t_0}, \mu_{t_0})$. By Theorem 3.9, $\eta_t = \mu_t$. Therefore, $I \underset{\neq}{\subset} \mu_{t_0} \subseteq \mu_t = \eta_t$.

Define an L-subset $\theta : R \to L$, as follows

$$\theta(x) = \begin{cases} t_0, & x \in I - \eta_{t_0} \\ \eta(x), & x \in \eta_{t_0} \cup (R - I) \end{cases}.$$

We show that $\eta \underset{\neq}{\subset} \theta \underset{\neq}{\subset} \mu$. For $x \in I - \eta_{t_0} \subseteq \mu_{t_0}$, by the definition of θ, we have $\eta(x) < t_0 = \theta(x) \le \mu(x)$. Also, for $x \in \eta_{t_0} \cup (R - I)$, we have $\eta(x) = \theta(x) \le \mu(x)$. Therefore $\eta \underset{\neq}{\subseteq} \theta \subseteq \mu$. For $x_0 \in \mu_{t_0} - I$, by the definition of θ, we have $\theta(x_0) = \eta(x_0)$. Since $x_0 \in \mu_{t_0}$ and $x_0 \notin I \underset{\neq}{\supset} \eta_{t_0}$, we have $\eta(x_0) < t_0 \le \mu(x_0)$. Thus $\eta \underset{\neq}{\subset} \theta \underset{\neq}{\subset} \mu$.

To show that θ is an ideal of μ, we show that each non-empty level subset θ_t is an ideal of the level subset μ_t. Suppose θ_t is a non-empty level subset. Consider the following cases.

Case (i) $t = t_0$. We show that $\theta_{t_0} = I$. Since $\theta(x) = t_0$, $\forall$ $x \in I - \eta_{t_0}$ and $\theta(x) = \eta(x) \ge t_0, \forall$ $x \in \eta_{t_0}$, we have $\theta(x) \ge t_0$, $\forall$ $x \in I$. Hence $I \subseteq \theta_{t_0}$. To the reverse inclusion, let $x_0 \notin I$. Then $x_0 \in R - I$ and $x_0 \notin \eta_{t_0}$. Therefore by the definition of θ, we have $\theta(x_0) = \eta(x_0) < t_0$ and thus $x_0 \notin \theta_{t_0}$. Since I is an ideal of the level subset μ_{t_0}, θ_{t_0} is an ideal of μ_{t_0}.

Case (ii) $t > t_0$. We show that $\theta_t = \eta_t$. Since $\eta \subseteq \theta$, we have $\eta_t \subseteq \theta_t$. Suppose $x_0 \in \theta_t$. Then $\theta(x_0) \ge t > t_0$. Hence by the definition of θ, $x_0 \in \eta_{t_0} \cup (R - I)$ and thus $\eta(x_0) = \theta(x_0) \ge t$. So that $x_0 \in \eta_t$ and thus $\theta_t \subseteq \eta_t$. Since η is an ideal of μ, the level subset η_t is an ideal of level subset μ_t. Thus θ_t is an ideal of μ_t.

Case (iii) $t < t_0$ and there is no $t_1 \in Im\eta$ such that $t \le t_1 < t_0$. By the definition of θ, there is no $t_1 \in Im\theta$ such that $t \le t_1 < t_0$. Thus $\theta_t = \theta_{t_0} = I$. Now we show that there is no $t' \in Im\mu$ such that $t \le t' < t_0$. If possible, there exists $t' \in Im\,\mu$ such that $t \le t' < t_0$. Then $\mu_{t_0} \underset{\neq}{\subset} \mu_{t'} \subseteq \mu_t$. Therefore $(\eta_t, \mu_t) \ne (\eta_{t_0}, \mu_{t_0})$. Now by Theorem 2.9, $\eta_t = \mu_t$ and hence $\eta_{t_0} \underset{\neq}{\subset} I \underset{\neq}{\subset} \mu_{t_0} \subseteq \mu_t = \eta_t$. That is, $\eta_{t_0} \underset{\neq}{\subset} \eta_t$.

Consequently there exists $t_1 \in \operatorname{Im}\eta$ such that $t \le t_1 < t_0$, which is a contradiction. Therefore, there is no $t' \in \operatorname{Im}\mu$ such that $t \le t' < t_0$ and hence, $\mu_t = \mu_{t_0}$. Moreover, as $\theta_t = I$ and I is an ideal of μ_{t_0}, the level subset θ_t is an ideal of μ_t.

Case (iv) $t < t_0$ and there exists $t_1 \in \operatorname{Im}\eta$ such that $t \le t_1 < t_0$. We show that $\theta_t = \eta_t$. In view of the fact that $\eta \subseteq \theta$, it is sufficient to show that $\theta_t \subseteq \eta_t$. Let $x_0 \in \theta_t$. Then $\theta(x_0) \ge t$. But either $\theta(x_0) \ge t_0$ or $\theta(x_0) < t_0$. Now, if $\theta(x_0) \ge t_0$, then $x_0 \in \theta_{t_0} = I$. As it is shown in the beginning of the proof that for every $t_1 \in \operatorname{Im}\eta$ with $t_1 < t_0$, we have $I \underset{\neq}{\subset} \eta_{t_1}$, therefore $x_0 \in \theta_{t_0} = I \underset{\neq}{\subset} \eta_{t_1} \subseteq \eta_t$. And, if $\theta(x_0) < t_0$, then by the definition of θ, we have $x_0 \in \eta_{t_0} \cup (R - I)$ and hence $\eta(x_0) = \theta(x_0) \ge t$. Thus $x_0 \in \eta_t$. Therefore, $\theta_t \subseteq \eta_t$. Since η is an ideal of μ, the level subset η_t is an ideal of the level subset μ_t. That is, θ_t is an ideal of μ_t.

Now, θ is an ideal of μ. This contradicts that η is a maximal ideal of μ. Thus η_{t_0} is a maximal ideal of μ_{t_0}. ν

The following result displays the role of the range set Imμ of an L-ring L(μ,R) in the construction of maximal ideals of μ.

Corollary 2.11. *Suppose L(μ,R) is an L-ring and η is a maximal ideal of μ with* $\eta(0) = \mu(0)$. *Then there exists* $t_0 \in \operatorname{Im}\mu$ *such that* η_{t_0} *is a maximal ideal of* μ_{t_0} *and* $\eta_t = \mu_t$, $\forall t \in \operatorname{Im}\mu - \{t_0\}$.

Proof. Since $\eta \underset{\neq}{\subset} \mu$, we have $\eta_t \subseteq \mu_t$, $\forall$ $t \in L$ and there exists $x_0 \in R$ such that $\eta(x_0) < \mu(x_0) = t_0$ (say). Thus $\eta_{t_0} \underset{\neq}{\subseteq} \mu_{t_0}$. Now $\mu_t \ne \mu_{t_0}$, $\forall$ $t \in \operatorname{Im}\mu - \{t_0\}$. Thus

$(\eta_t,\mu_t)\neq(\eta_{t_0},\mu_{t_0})$, $\forall$ $t\in \operatorname{Im}\mu-\{t_0\}$. In view of Theorem 2.9, $\eta_t=\mu_t$, $\forall$ $t\in \operatorname{Im}\mu-\{t_0\}$. Again by Theorem 3.10, there exists a pair (η_{t_1},μ_{t_1}) such that η_{t_1} is a maximal ideal of μ_{t_1} and for all other pairs (η_t,μ_t), $\eta_t=\mu_t$. Since $\eta_t=\mu_t$, $\forall$ $t\in \operatorname{Im}\mu-\{t_0\}$, we have $(\eta_{t_1},\mu_{t_1})\neq(\eta_t,\mu_t)$, $\forall$ $t\in \operatorname{Im}\mu-\{t_1\}$.

Now we show that $(\eta_{t_1},\mu_{t_1})=(\eta_{t_0},\mu_{t_0})$. Suppose $(\eta_{t_1},\mu_{t_1})\neq(\eta_{t_0},\mu_{t_0})$. Then by Theorem 2.10, $\eta_{t_0}=\mu_{t_0}$, which is a contradiction. Hence, $(\eta_{t_1},\mu_{t_1})=(\eta_{t_0},\mu_{t_0})$. Thus $\eta_{t_0}=\eta_{t_1}$ and $\mu_{t_0}=\mu_{t_1}$. Consequently, η_{t_0} is a maximal ideal of μ_{t_0}. ν

The converse of the above result is however, not true.

Example 2.12. Let L be a five elements chain, $t_3<t_2<t_1<t_0<t_4$, and R be the ring of integers. Define an L-ring L(μ,R) as follows

$$\mu(x)=\begin{cases} t_4, & x\in(0)\\ t_0, & x\in(2^2)-(0)\\ t_3, & x\in R-(2^2)\end{cases}$$

Define an L-subset $\eta:R\to L$ by

$$\eta(x)=\begin{cases} t_4 & , \quad x\in(0)\\ t_0 & , \quad x\in(2^3)-(0)\\ t_2 & , \quad x\in(2^2)-(2^3)\\ t_3 & , \quad x\in R-(2^2)\end{cases}$$

Clearly $\operatorname{Im}\mu=\{t_0,t_3,t_4\}$, $\mu_{t_3}=R$, $\mu_{t_0}=(2^2)$, $\mu_{t_4}=(0)$, $\eta_{t_3}=R$, $\eta_{t_0}=(2^3)$, $\eta_{t_4}=(0)$ and η is an ideal of μ such that η_{t_0} is a maximal ideal of μ_{t_0} with $\eta_t=\mu_t$, $\forall$ $t\in \operatorname{Im}\mu-\{t_0\}$. Define an L-subset $\theta:R\to L$ as follows

$$\theta(x) = \begin{cases} t_0 & , \quad x \in (0) \\ t_1 & ; \quad x \in (2^3) - (0) \\ t_3 & , \quad x \in (2^2) - (2^3) \\ t_4 & , \quad x \in R - (2^2) \end{cases}$$

The L-subset θ is an ideal of μ such that $\eta \underset{\neq}{\subset} \theta \underset{\neq}{\subset} \mu$. Hence η is not a maximal ideal of μ. ν

The following theorem shows that for a maximal ideal η of an L-ring L(μ,R), the maximality of a level subset of η in the corresponding level subset of μ implies the hypothesis of the Corollary 2.11.

Theorem 2.13. *Suppose L(μ,R) is an L-ring and η is a maximal ideal of μ such that* η_{t_0} *is a maximal ideal of* μ_{t_0}*, for some* $t_0 \in \operatorname{Im}\mu$ *. Then* $\eta(0) = \mu(0)$ *Also* $\eta_t = \mu_t$, $\forall t \in \operatorname{Im}\mu - \{t_0\}$.

Proof. By Corollary 2.7, either $\eta(0) = \mu(0)$ or $\mu(0)$ is a cover of η(0). Suppose μ(0) is cover of η(0). Define an L-subset $\theta : R \to L$, as follows

$$\theta(x) = \begin{cases} \mu(0) & , \quad x \in (0) \\ \eta(x) & , \quad x \in R - (0) \end{cases}$$

Now $\eta(0) < \mu(0) = \theta(0)$. For $x \in R - (0)$, $\theta(x) = \eta(x) \le \mu(x)$. Thus $\eta \underset{\neq}{\subset} \theta \subseteq \mu$. Since η_{t_0} is a maximal ideal of μ_{t_0} for some $t_0 \in \operatorname{Im}\mu$, we have $\eta_{t_0} \underset{\neq}{\subset} \mu_{t_0}$. Thus there exists $x_0 \neq 0$ in μ_{t_0} such that $x_0 \notin \eta_{t_0}$. Hence $\theta(x_0) = \eta(x_0) < t_0 \le \mu(x_0)$. Thus $\eta \underset{\neq}{\subset} \theta \underset{\neq}{\subset} \mu$

Now

$$\theta_t = \begin{cases} (0) & , \quad t = \mu(0) \\ \eta_t & , \quad t \le \eta(0) \end{cases}$$

Since η is an ideal of μ, η_t is an ideal of μ_t, $\forall\ t \leq \eta(0)$. Also (0) is an ideal of μ_t, $\forall\ t \leq \mu(0)$. θ is ideal of μ. This contradicts that η is a maximal ideal of μ. Hence $\eta(0) = \mu(0)$. By Theorem 2.11, $\eta_t = \mu_t$, $\forall\ t \in \operatorname{Im}\mu - \{t_0\}$. ν

In the following theorem we provide sufficient conditions for an ideal η to be maximal in an L-ring L(μ,R).

Theorem 2.14. *Suppose L(μ,R) is an L-ring and η is an ideal of μ such that*

(i) η_{t_0} *is a maximal ideal of* μ_{t_0}, *for some* $t_0 \in \operatorname{Im}\mu$,

(ii) $\eta_t = \mu_t$, $\forall\ t \in \operatorname{Im}\mu - \{t_0\}$,

(iii) t_0 is cover of t_1 for some $t_1 \in \operatorname{Im}\mu$,

Then η is a maximal ideal of μ.

In order to prove the above theorem, we state the following

Lemma 2.15. Suppose $\theta, \mu \in L^R$ such that $\theta \subseteq \mu$ and $\theta_t = \mu_t$, $\forall\ t \in \operatorname{Im}\mu$. Then $\theta = \mu$.

Proof of the Theorem. Suppose θ is an ideal of μ such that $\eta \underset{\neq}{\subset} \theta \subseteq \mu$. Then $\eta_t \subseteq \theta_t \subseteq \mu_t$, $\forall\ t \in L$. Since $\eta_t = \mu_t$, $\forall\ t \in \operatorname{Im}\mu - \{t_0\}$, we have $\theta_t = \mu_t\ \forall\ t \in \operatorname{Im}\mu - \{t_0\}$. Now we show that $\theta_{t_0} = \mu_{t_0}$. Since $\eta \underset{\neq}{\subset} \theta \subseteq \mu$, there exists $x_0 \in R$ such that $\eta(x_0) < \theta(x_0) \leq \mu(x_0)$. Write $\theta(x_0) = t'$ and $\mu(x_0) = t_2$. Then $\eta(x_0) < \theta(x_0) = t' \leq \mu(x_0) = t_2$. Thus $\eta_{t_2} \underset{\neq}{\subset} \mu_{t_2}$, where $t_2 \in \operatorname{Im}\mu$. Hence by the hypotheses (i) and (ii), $t_2 = t_0$. Thus $t' \leq t_0$. Suppose that $t' < t_0$. Then by the hypothesis (iii) $t' \leq t_1 < t_0$, for some $t_1 \in \operatorname{Im}\mu$. Now, $\eta(x_0) < \theta(x_0) = t' \leq t_1 < t_0 = \mu(x_0)$, imply that $\eta_{t_1} \underset{\neq}{\subset} \mu_{t_1}$. This contradicts (ii). Thus we must have $t' = t_0$.

Consequently, $\eta(x_0) < \theta(x_0) = t_0$. Hence $\eta_{t_0} \underset{\neq}{\subset} \theta_{t_0} \subseteq \mu_{t_0}$. Since η_{t_0} is a maximal ideal of μ_{t_0} and θ_{t_0} is an ideal of μ_{t_0}, we have $\theta_{t_0} = \mu_{t_0}$. Thus $\theta_t = \mu_t$, $\forall\ t \in \operatorname{Im}\mu$. Now using Lemma 3.15, we get $\theta = \mu$. Hence η is a maximal ideal of μ. ν

Consider the following example.

Example 2.16. Let L be a five elements chain $t_3 < t_2 < t_1 < t_0 < t_4$ and R be the ring of integers. Define an L-ring L(μ,R) as follows :

$$\mu(x) = \begin{cases} t_4 & , \quad x \in (0) \\ t_0 & ; \quad x \in (2^2) - (0) \\ t_3 & , \quad x \in R - (2^2) \end{cases}.$$

Define an L-subset $\eta: R \to L$ by

$$\eta(x) = \begin{cases} t_4 & , \quad x \in (0) \\ t_0 & ; \quad x \in (2^3) - (0) \\ t_1 & , \quad x \in (2^2) - (2^3) \\ t_3 & , \quad x \in R - (2^2) \end{cases}$$

Here η is an ideal of μ and $\operatorname{Im}\mu = \{t_3, t_0, t_4\}$. Moreover t_0 is a cover of t_1 and $t_1 \notin \operatorname{Im}\mu$. It can be shown that η is a maximal ideal of μ. ν

The above example shows that the conditions of Theorem 2.14 are only sufficient but not necessary for an ideal η to be maximal ideal of μ.

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